HW #3

IS

$\boxed{\mathbf{F_2} \text{ is a gradient; } \mathbf{F_1} \text{ is a curl}}$ $\boxed{U_2 = \frac{1}{2}(x^2+y^2+z^2)}$ would do ($\mathbf{F_2} = \nabla U_2$).

For $\mathbf{A_1}$, we want $\left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right) = \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) = 0$; $\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = x^2$. $A_y = \frac{x^3}{3}$, $A_x = A_z = 0$ would do it.

$\boxed{\mathbf{A_1} = \frac{1}{3}x^2\,\hat{\mathbf{y}}}$ ($\mathbf{F_1} = \nabla\times\mathbf{A_1}$). (But these are not unique.)

(b) $\nabla\cdot\mathbf{F_3} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$; $\nabla\times\mathbf{F_3} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \hat{\mathbf{x}}(x-x) + \hat{\mathbf{y}}(y-y) + \hat{\mathbf{z}}(z-z) = 0$

So $\mathbf{F_3}$ can be written as the gradient of a scalar ($\mathbf{F_3} = \nabla U_3$) and as the curl of a vector ($\mathbf{F_3} = \nabla\times\mathbf{A_3}$). In fact, $\boxed{U_3 = xyz}$ does the job. For the vector potential, we have

$$\left\{\begin{array}{llll} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = yz, & \text{which suggests} & A_z = \frac{1}{4}y^2z + f(x,z); & A_y = -\frac{1}{4}yz^2 + g(x,y) \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = xz, & \text{suggesting} & A_x = \frac{1}{4}z^2x + h(x,y); & A_z = -\frac{1}{4}zx^2 + j(y,z) \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = xy, & \text{so} & A_y = \frac{1}{4}x^2y + k(y,z); & A_x = -\frac{1}{4}xy^2 + l(x,y) \end{array}\right\}$$

Putting this all together: $\boxed{\mathbf{A_3} = \frac{1}{4}\left\{x(z^2-y^2)\,\hat{\mathbf{x}} + y(x^2-z^2)\,\hat{\mathbf{y}} + z(y^2-x^2)\,\hat{\mathbf{z}}\right\}}$ (again, not unique).

Problem 1.50

(d) ⇒ (a): $\nabla\times\mathbf{F} = \nabla\times(-\nabla U) = 0$ (Eq. 1.44 – curl of gradient is always zero).

(a) ⇒ (c): $\oint \mathbf{F}\cdot d\mathbf{l} = \int(\nabla\times\mathbf{F})\cdot d\mathbf{a} = 0$ (Eq. 1.57–Stokes' theorem).

(c) ⇒ (b): $\int_{\mathbf{a}\ I}^{\mathbf{b}} \mathbf{F}\cdot d\mathbf{l} - \int_{\mathbf{a}\ II}^{\mathbf{b}} \mathbf{F}\cdot d\mathbf{l} = \int_{\mathbf{a}\ I}^{\mathbf{b}} \mathbf{F}\cdot d\mathbf{l} + \int_{\mathbf{b}\ II}^{\mathbf{a}} \mathbf{F}\cdot d\mathbf{l} = \oint \mathbf{F}\cdot d\mathbf{l} = 0$, so

$$\int_{\mathbf{a}\ I}^{\mathbf{b}} \mathbf{F}\cdot d\mathbf{l} = \int_{\mathbf{a}\ II}^{\mathbf{b}} \mathbf{F}\cdot d\mathbf{l}.$$

(b) ⇒ (c): same as (c) ⇒ (b), only in reverse; (c) ⇒ (a): same as (a)⇒ (c).

Problem 1.51

(d) ⇒ (a): $\nabla\cdot\mathbf{F} = \nabla\cdot(\nabla\times\mathbf{W}) = 0$ (Eq 1.46—divergence of curl is always zero).

(a) ⇒ (c): $\oint \mathbf{F}\cdot d\mathbf{a} = \int(\nabla\cdot\mathbf{F})\,d\tau = 0$ (Eq. 1.56—divergence theorem).

(c) ⇒ (b): $\int_I \mathbf{F}\cdot d\mathbf{a} - \int_{II} \mathbf{F}\cdot d\mathbf{a} = \oint \mathbf{F}\cdot d\mathbf{a} = 0$, so

$$\int_I \mathbf{F}\cdot d\mathbf{a} = \int_{II} \mathbf{F}\cdot d\mathbf{a}.$$

(*Note:* sign change because for $\oint \mathbf{F}\cdot d\mathbf{a}$, da is *outward*, whereas for surface II it is *inward.*)

(b) ⇒ (c): same as (c) ⇒ (b), in reverse; (c)⇒ (a): same as (a)⇒ (c) .

Problem 1.52

In Prob. 1.15 we found that $\nabla\cdot\mathbf{v}_a = 0$; in Prob. 1.18 we found that $\nabla\times\mathbf{v}_c = 0$. So

$\boxed{\mathbf{v}_c \text{ can be written as the gradient of a scalar; } \mathbf{v}_a \text{ can be written as the curl of a vector.}}$

(a) To find t:

(1) $\frac{\partial t}{\partial x} = y^2 \Rightarrow t = y^2x + f(y,z)$

(2) $\frac{\partial t}{\partial y} = (2xy + z^2)$

(3) $\frac{\partial t}{\partial z} = 2yz$

From (1) & (3) we get $\frac{\partial f}{\partial z} = 2yz \Rightarrow f = yz^2 + g(y) \Rightarrow t = y^2x + yz^2 + g(y)$, so $\frac{\partial t}{\partial y} = 2xy + z^2 + \frac{\partial g}{\partial y} = 2xy + z^2$ (from (2)) $\Rightarrow \frac{\partial g}{\partial y} = 0$. We may as well pick $g = 0$; then $\boxed{t = xy^2 + yz^2.}$

(b) To find **W**: $\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z} = x^2$; $\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x} = 3z^2x$; $\frac{\partial W_y}{\partial x} - \frac{\partial W_x}{\partial y} = -2xz$.
Pick $W_x = 0$; then

$$\begin{aligned}\frac{\partial W_z}{\partial x} &= -3xz^2 \Rightarrow W_z = -\frac{3}{2}x^2z^2 + f(y,z)\\ \frac{\partial W_y}{\partial x} &= -2xz \Rightarrow W_y = -x^2z + g(y,z).\end{aligned}$$

$\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z} = \frac{\partial f}{\partial y} + x^2 - \frac{\partial g}{\partial z} = x^2 \Rightarrow \frac{\partial f}{\partial y} - \frac{\partial g}{\partial z} = 0$. May as well pick $f = g = 0$.

$\boxed{\mathbf{W} = -x^2z\,\hat{\mathbf{y}} - \frac{3}{2}x^2z^2\,\hat{\mathbf{z}}.}$

Check: $\nabla\times\mathbf{W} = \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & -x^2z & -\frac{3}{2}x^2z^2\end{vmatrix} = \hat{\mathbf{x}}\,(x^2) + \hat{\mathbf{y}}\,(3xz^2) + \hat{\mathbf{z}}\,(-2xz).\checkmark$

You can add any gradient (∇t) to **W** without changing its curl, so this answer is far from unique. Some other solutions:

$\mathbf{W} = xz^3\,\hat{\mathbf{x}} - x^2z\,\hat{\mathbf{y}}$;

$\mathbf{W} = \left(2xyz + xz^3\right)\,\hat{\mathbf{x}} + x^2y\,\hat{\mathbf{z}}$;

$\mathbf{W} = xyz\,\hat{\mathbf{x}} - \frac{1}{2}x^2z\,\hat{\mathbf{y}} + \frac{1}{2}x^2\left(y - 3z^2\right)\,\hat{\mathbf{z}}$.

Probelm 1.53

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,r^2\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,r^2\cos\phi\right) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(-r^2\cos\theta\sin\phi\right)\\ &= \frac{1}{r^2}4r^3\cos\theta + \frac{1}{r\sin\theta}\cos\theta\,r^2\cos\phi + \frac{1}{r\sin\theta}\left(-r^2\cos\theta\cos\phi\right)\\ &= \frac{r\cos\theta}{\sin\theta}\left[4\sin\theta + \cos\phi - \cos\phi\right] = 4r\cos\theta.\end{aligned}$$

$$\begin{aligned}\int(\nabla\cdot\mathbf{v})\,d\tau &= \int(4r\cos\theta)r^2\sin\theta\,dr\,d\theta\,d\phi = 4\int_0^R r^3\,dr\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi\\ &= \left(R^4\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi R^4}{4}.}\end{aligned}$$

Surface consists of four parts:
(1) *Curved:* $d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$; $r = R$. $\mathbf{v}\cdot d\mathbf{a} = \left(R^2\cos\theta\right)\left(R^2\sin\theta\,d\theta\,d\phi\right)$.

$$\int\mathbf{v}\cdot d\mathbf{a} = R^4\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi = R^4\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi R^4}{4}.$$

(2) *Left:* $\mathbf{da} = -r\,dr\,d\theta\,\hat{\boldsymbol{\phi}}$; $\phi = 0$. $\mathbf{v}\cdot\mathbf{da} = \left(r^2\cos\theta\sin\phi\right)(r\,dr\,d\theta) = 0$. $\int \mathbf{v}\cdot\mathbf{da} = 0$.
(3) *Back:* $\mathbf{da} = r\,dr\,d\theta\,\hat{\boldsymbol{\phi}}$; $\phi = \pi/2$. $\mathbf{v}\cdot\mathbf{da} = \left(-r^2\cos\theta\sin\phi\right)(r\,dr\,d\theta) = -r^3\cos\theta\,dr\,d\theta$.

$$\int \mathbf{v}\cdot\mathbf{da} = \int_0^R r^3\,dr \int_0^{\pi/2} \cos\theta\,d\theta = -\left(\frac{1}{4}R^4\right)(+1) = -\frac{1}{4}R^4.$$

(4) *Bottom:* $\mathbf{da} = r\sin\,dr\,d\phi\,\hat{\boldsymbol{\theta}}$; $\theta = \pi/2$. $\mathbf{v}\cdot\mathbf{da} = \left(r^2\cos\phi\right)(r\,dr\,d\phi)$.

$$\int \mathbf{v}\cdot\mathbf{da} = \int_0^R r^3\,dr \int_0^{\pi/2} \cos\phi\,d\phi = \frac{1}{4}R^4.$$

Total: $\oint \mathbf{v}\cdot\mathbf{da} = \pi R^4/4 + 0 - \frac{1}{4}R^4 + \frac{1}{4}R^4 = \frac{\pi R^4}{4}$. ✓

Problem 1.54

$$\nabla\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay & bx & 0 \end{vmatrix} = \hat{\mathbf{z}}\,(b-a). \quad \text{So} \quad \int(\nabla\times\mathbf{v})\cdot\mathbf{da} = (b-a)\pi R^2.$$

$\mathbf{v}\cdot d\mathbf{l} = (ay\,\hat{\mathbf{x}} + bx\,\hat{\mathbf{y}})\cdot(dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) = ay\,dx + bx\,dy$; $x^2 + y^2 = R^2 \Rightarrow 2x\,dx + 2y\,dy = 0$, so $dy = -(x/y)\,dx$. So $\mathbf{v}\cdot d\mathbf{l} = ay\,dx + bx(-x/y)\,dx = \frac{1}{y}\left(ay^2 - bx^2\right)dx$.

For the "upper" semicircle, $y = \sqrt{R^2 - x^2}$, so $\mathbf{v}\cdot d\mathbf{l} = \frac{a(R^2-x^2)-bx^2}{\sqrt{R^2-x^2}}\,dx$.

$$\begin{aligned}\int \mathbf{v}\cdot d\mathbf{l} &= \int_R^{-R} \frac{aR^2 - (a+b)x^2}{\sqrt{R^2-x^2}}\,dx = \left\{ aR^2\sin^{-1}\left(\frac{x}{R}\right) - (a+b)\left[-\frac{x}{2}\sqrt{R^2-x^2} + \frac{R^2}{2}\sin^{-1}\left(\frac{x}{R}\right)\right]\right\}\Bigg|_{+R}^{-R} \\ &= \frac{1}{2}R^2(a-b)\sin^{-1}(x/R)\Big|_{+R}^{-R} = \frac{1}{2}R^2(a-b)\left(\sin^{-1}(-1) - \sin^{-1}(+1)\right) = \frac{1}{2}R^2(a-b)\left(-\frac{\pi}{2} - \frac{\pi}{2}\right) \\ &= \frac{1}{2}\pi R^2(b-a).\end{aligned}$$

And the same for the lower semicircle (y changes sign, but the limits on the integral are reversed) so $\oint \mathbf{v}\cdot d\mathbf{l} = \pi R^2(b-a)$. ✓

Problem 1.55

(1) $x = z = 0$; $dx = dz = 0$; $y : 0 \to 1$. $\mathbf{v}\cdot d\mathbf{l} = (y + 3x)\,dy = y\,dy$.

$$\int_0^1 \mathbf{v}\cdot d\mathbf{l} = \int_0^1 y\,dy = \frac{1}{2}.$$

(2) $x = 0$; $z = 2 - 2y$; $dz = -2\,dy$; $y : 1 \to 0$. $\mathbf{v}\cdot d\mathbf{l} = (y + 3x)\,dy + 6\,dz = y\,dy - 12\,dy = (y - 12)\,dy$.

$$\int \mathbf{v}\cdot d\mathbf{l} = \int_1^0 (y-12)\,dy = -\left(\frac{1}{2} - 12\right) = -\frac{1}{2} + 12.$$

(3) $x = y = 0$; $dx = dy = 0$; $z : 2 \to 0$. $\mathbf{v}\cdot d\mathbf{l} = 6\,dz$;

$$\int \mathbf{v}\cdot d\mathbf{l} = \int_2^0 6\,dz = -12.$$

Total: $\oint \mathbf{v}\cdot d\mathbf{l} = \frac{1}{2} - \frac{1}{2} + 12 - 12 = \boxed{0.}$
Meanwhile, Stokes' thereom says $\oint \mathbf{v}\cdot d\mathbf{l} = \int(\nabla\times\mathbf{v})\cdot d\mathbf{a}$. Here $d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}$, so all we need is $(\nabla\times\mathbf{v})_x = \frac{\partial}{\partial y}(6) - \frac{\partial}{\partial z}(y+3x) = 0$. Therefore $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0$. ✓

Problem 1.56

Start at the origin.

(1) $\theta = \frac{\pi}{2}$, $\phi = 0$; $r: 0\to 1$. $\mathbf{v}\cdot d\mathbf{l} = \left(r\cos^2\theta\right)(dr) = 0$. $\int \mathbf{v}\cdot d\mathbf{l} = 0$.

(2) $r = 1$, $\theta = \frac{\pi}{2}$; $\phi: 0 \to \pi/2$. $\mathbf{v}\cdot d\mathbf{l} = (3r)(r\sin\theta\, d\phi) = 3\,d\phi$. $\int \mathbf{v}\cdot d\mathbf{l} = 3\int_0^{\pi/2} d\phi = \frac{3\pi}{2}$.

(3) $\phi = \frac{\pi}{2}$; $r\sin\theta = y = 1$, so $r = \frac{1}{\sin\theta}$, $dr = \frac{-1}{\sin^2\theta}\cos\theta\, d\theta$, $\theta: \frac{\pi}{2}\to\frac{\pi}{4}$.

$$\begin{aligned}
\mathbf{v}\cdot d\mathbf{l} &= \left(r\cos^2\theta\right)(dr) - (r\cos\theta\sin\theta)(r\,d\theta) = \frac{\cos^2\theta}{\sin\theta}\left(-\frac{\cos\theta}{\sin^2\theta}\right) d\,theta - \frac{\cos\theta\sin\theta}{\sin^2\theta}\,d\theta \\
&= -\left(\frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta}{\sin\theta}\right) d\theta = -\frac{\cos\theta}{\sin\theta}\left(\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta}\right) d\theta = -\frac{\cos\theta}{\sin^3\theta}\,d\theta.
\end{aligned}$$

Therefore

$$\int \mathbf{v}\cdot d\mathbf{l} = -\int_{\pi/2}^{\pi/4} \frac{\cos\theta}{\sin^3\theta}\,d\theta = \left.\frac{1}{2\sin^2\theta}\right|_{\pi/2}^{\pi/4} = \frac{1}{2\cdot(1/2)} - \frac{1}{2\cdot(1)} = 1 - \frac{1}{2} = \frac{1}{2}.$$

(4) $\theta = \frac{\pi}{4}$, $\phi = \frac{\pi}{2}$; $r: \sqrt{2}\to 0$. $\mathbf{v}\cdot d\mathbf{l} = \left(r\cos^2\theta\right)(dr) = \frac{1}{2}r\,dr$.

$$\int \mathbf{v}\cdot d\mathbf{l} = \frac{1}{2}\int_{\sqrt{2}}^{0} r\,dr = \left.\frac{1}{2}\frac{r^2}{2}\right|_{\sqrt{2}}^{0} = -\frac{1}{4}\cdot 2 = -\frac{1}{2}.$$

Total:

$$\oint \mathbf{v}\cdot d\mathbf{l} = 0 + \frac{3\pi}{2} + \frac{1}{2} - \frac{1}{2} = \boxed{\frac{3\pi}{2}}.$$

Stokes' theorem says this should equal $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a}$

$$\begin{aligned}
\nabla\times\mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, 3r) - \frac{\partial}{\partial\phi}(-r\sin\theta\cos\theta)\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(r\cos^2\theta\right) - \frac{\partial}{\partial r}(r3r)\right]\hat{\boldsymbol{\theta}} \\
&\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(-rr\cos\theta\sin\theta) - \frac{\partial}{\partial\theta}\left(r\cos^2\theta\right)\right]\hat{\boldsymbol{\phi}} \\
&= \frac{1}{r\sin\theta}[3r\cos\theta]\,\hat{\mathbf{r}} + \frac{1}{r}[-6r]\,\hat{\boldsymbol{\theta}} + \frac{1}{r}[-2r\cos\theta\sin\theta + 2r\cos\theta\sin\theta]\,\hat{\boldsymbol{\phi}} \\
&= 3\cot\theta\,\hat{\mathbf{r}} - 6\,\hat{\boldsymbol{\theta}}.
\end{aligned}$$

(1) *Back face:* $d\mathbf{a} = -r\,dr\,d\theta\,\hat{\boldsymbol{\phi}}$; $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0$. $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0$.

(2) *Bottom:* $d\mathbf{a} = -r\sin\theta\,dr\,d\phi\,\hat{\boldsymbol{\theta}}$; $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 6r\sin\theta\,dr\,d\phi$. $\theta = \frac{\pi}{2}$, so $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 6r\,dr\,d\phi$

$$\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = \int_0^1 6r\,dr\int_0^{\pi/2} d\phi = 6\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{2}. \quad ✓$$

Problem 1.57

$\mathbf{v}\cdot d\mathbf{l} = y\,dz$.

(1) *Left side:* $z = a - x$; $dz = -dx$; $y = 0$. Therefore $\int \mathbf{v}\cdot d\mathbf{l} = 0$.

(2) *Bottom:* $dz = 0$. Therefore $\int \mathbf{v}\cdot d\mathbf{l} = 0$.

(3) *Back:* $z = a - \frac{1}{2}y$; $dz = -1/2\,dy$; $y : 2a \to 0$. $\int \mathbf{v}\cdot d\mathbf{l} = \int_{2a}^{0} y\left(-\frac{1}{2}dy\right) = -\frac{1}{2}\frac{y^2}{2}\Big|_{2a}^{0} = \frac{4a^2}{4} = \boxed{a^2.}$

Meanwhile, $\nabla\times\mathbf{v} = \hat{\mathbf{x}}$, so $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a}$ is the projection of this surface on the xy plane $= \frac{1}{2}\cdot a\cdot 2a = a^2$. ✓

Problem 1.58

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^2\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, 4r^2\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(r^2\tan\theta\right)\\ &= \frac{1}{r^2}4r^3\sin\theta + \frac{1}{r\sin\theta}4r^2\left(\cos^2\theta - \sin^2\theta\right) = \frac{4r}{\sin\theta}\left(\sin^2\theta + \cos^2\theta - \sin^2\theta\right)\\ &= 4r\frac{\cos^2\theta}{\sin\theta}.\end{aligned}$$

$$\begin{aligned}\int(\nabla\cdot\mathbf{v})\,d\tau &= \int\left(4r\frac{\cos^2\theta}{\sin\theta}\right)\left(r^2\sin\theta\,dr\,d\theta\,d\phi\right) = \int_0^R 4r^3\,dr\int_0^{\pi/6}\cos^2\theta\,d\theta\int_0^{2\pi}d\phi = \left(R^4\right)(2\pi)\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]\Big|_0^{\pi/6}\\ &= 2\pi R^4\left(\frac{\pi}{12} + \frac{\sin 60^\circ}{4}\right) = \frac{\pi R^4}{6}\left(\pi + 3\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\pi R^4}{12}\left(2\pi + 3\sqrt{3}\right).}\end{aligned}$$

Surface consists of two parts:

(1) *The ice cream:* $r = R$; $\phi : 0 \to 2\pi$; $\theta : 0 \to \pi/6$; $d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$; $\mathbf{v}\cdot d\mathbf{a} = \left(R^2\sin\theta\right)\left(R^2\sin\theta\,d\theta\,d\phi\right) = R^4\sin^2\theta\,d\theta\,d\phi$.

$$\int\mathbf{v}\cdot d\mathbf{a} = R^4\int_0^{\pi/6}\sin^2\theta\,d\theta\int_0^{2\pi}d\phi = \left(R^4\right)(2\pi)\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/6} = 2\pi R^4\left(\frac{\pi}{12} - \frac{1}{4}\sin 60^\circ\right) = \frac{\pi R^4}{6}\left(\pi - 3\frac{\sqrt{3}}{2}\right)$$

(2) *The cone:* $\theta = \frac{\pi}{6}$; $\phi : 0 \to 2\pi$; $r : 0 \to R$; $d\mathbf{a} = r\sin\theta\,d\phi\,dr\,\hat{\boldsymbol{\theta}} = \frac{\sqrt{3}}{2}r\,dr\,d\phi\,\hat{\boldsymbol{\theta}}$; $\mathbf{v}\cdot d\mathbf{a} = \sqrt{3}\,r^3\,dr\,d\phi$

$$\int\mathbf{v}\cdot d\mathbf{a} = \sqrt{3}\int_0^R r^3\,dr\int_0^{2\pi}d\phi = \sqrt{3}\cdot\frac{R^4}{4}\cdot 2\pi = \frac{\sqrt{3}}{2}\pi R^4.$$

Therefore $\int\mathbf{v}\cdot d\mathbf{a} = \frac{\pi R^4}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \sqrt{3}\right) = \frac{\pi R^4}{12}\left(2\pi + 3\sqrt{3}\right)$. ✓

Problem 1.59

(a) Corollary 2 says $\oint(\nabla T)\cdot d\mathbf{l} = 0$. Stokes' theorem says $\oint(\nabla T)\cdot d\mathbf{l} = \int[\nabla\times(\nabla T)]\cdot d\mathbf{a}$. So $\int[\nabla\times(\nabla T)]\cdot d\mathbf{a} = 0$, and since this is true for *any* surface, the integrand must vanish: $\nabla\times(\nabla T) = 0$, confirming Eq. 1.44.

(e) Let $T = \mathbf{c}\cdot\mathbf{r}$, and use product rule #4: $\nabla T = \nabla(\mathbf{c}\cdot\mathbf{r}) = \mathbf{c}\times(\nabla\times\mathbf{r}) + (\mathbf{c}\cdot\nabla)\mathbf{r}$. But $\nabla\times\mathbf{r} = 0$, and $(\mathbf{c}\cdot\nabla)\mathbf{r} = (c_x\frac{\partial}{\partial x} + c_y\frac{\partial}{\partial y} + c_z\frac{\partial}{\partial z})(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} = z\,\hat{\mathbf{z}}) = c_x\,\hat{\mathbf{x}} + c_y\,\hat{\mathbf{y}} + c_z\,\hat{\mathbf{z}} = \mathbf{c}$. So Prob. 1.60(e) says

$$\oint T\,d\mathbf{l} = \oint(\mathbf{c}\cdot\mathbf{r})\,d\mathbf{l} = -\int(\nabla T)\times d\mathbf{a} = -\int \mathbf{c}\times d\mathbf{a} = -\mathbf{c}\times\int d\mathbf{a} = -\mathbf{c}\times\mathbf{a} = \mathbf{a}\times\mathbf{c}. \qquad \text{qed}$$

Problem 1.62

(1)

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(r) = \boxed{\frac{1}{r^2}.}$$

For a sphere of radius R:

$$\left.\begin{aligned}\int \mathbf{v}\cdot d\mathbf{a} &= \int\left(\tfrac{1}{R}\,\hat{\mathbf{r}}\right)\cdot\left(R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}\right) = R\int\sin\theta\,d\theta\,d\phi = 4\pi R.\\ \int(\nabla\cdot\mathbf{v})\,d\tau &= \int\left(\tfrac{1}{r^2}\right)\left(r^2\sin\theta\,dr\,d\theta\,d\phi\right) = \left(\int_0^R dr\right)\left(\int\sin\theta\,d\theta\,d\phi\right) = 4\pi R.\end{aligned}\right\}\ \text{So divergence theorem checks.}$$

Evidently there is *no* delta function at the origin.

$$\nabla\times(r^n\,\hat{\mathbf{r}}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^n\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^{n+2}\right) = \frac{1}{r^2}(n+2)r^{n+1} = \boxed{(n+2)r^{n-1}}$$

(except for $n = -2$, for which we already know (Eq. 1.99) that the divergence is $4\pi\delta^3(\mathbf{r})$).

(2) *Geometrically,* it should be zero. Likewise, the curl in the spherical coordinates obviously gives $\boxed{\text{zero.}}$ To be certain there is no lurking delta function here, we integrate over a sphere of radius R, using Prob. 1.60(b): If $\nabla\times(r^n\,\hat{\mathbf{r}}) = 0$, then $\int(\nabla\times\mathbf{v})\,d\tau = 0 \overset{?}{=} -\oint \mathbf{v}\times d\mathbf{a}$. But $\mathbf{v} = r^n\,\hat{\mathbf{r}}$ and $d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$ are both in the $\hat{\mathbf{r}}$ directions, so $\mathbf{v}\times d\mathbf{a} = 0$. ✓

Chapter 2

Electrostatics

Problem 2.1

(a) $\boxed{\text{Zero.}}$

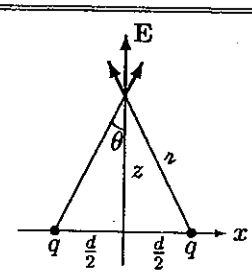
(b) $\boxed{F = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2},}$ where r is the distance from center to each numeral. **F** points *toward* the missing q. *Explanation:* by superposition, this is equivalent to (a), with an extra $-q$ at 6 o'clock—since the force of all twelve is zero, the net force is that of $-q$ only.

(c) $\boxed{\text{Zero.}}$

(d) $\boxed{\frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2},}$ pointing toward the missing q. Same reason as (b). Note, however, that if you explained (b) as a cancellation in pairs of opposite charges (1 o'clock against 7 o'clock; 2 against 8, etc.), with one unpaired q doing the job, then you'll need a *different* explanation for (d).

Problem 2.2

(a) "Horizontal" components cancel. Net vertical field is: $E_z = \frac{1}{4\pi\epsilon_0} 2\frac{q}{\imath^2}\cos\theta$.



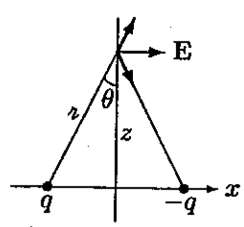
Here $\imath^2 = z^2 + \left(\frac{d}{2}\right)^2$; $\cos\theta = \frac{z}{\imath}$, so $\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2qz}{\left(z^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}}\,\hat{\mathbf{z}}.}$

When $z \gg d$ you're so far away it just looks like a single charge $2q$; the field should reduce to $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2q}{z^2}\,\hat{\mathbf{z}}$. And it *does* (just set $d \to 0$ in the formula).

(b) This time the "vertical" components cancel, leaving

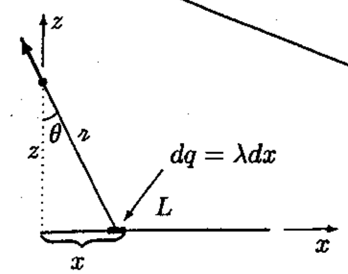
$\mathbf{E} = \frac{1}{4\pi\epsilon_0} 2\frac{q}{\imath^2}\sin\theta\,\hat{\mathbf{x}}$, or

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{qd}{\left(z^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}}\,\hat{\mathbf{x}}.}$$



From far away, ($z \gg d$), the field goes like $\mathbf{E} \approx \frac{1}{4\pi\epsilon_0}\frac{qd}{z^3}\,\hat{\mathbf{z}}$, which, as we shall see, is the field of a *dipole*. (If we set $d \to 0$, we get $\mathbf{E} = 0$, as is appropriate; to the extent that this configuration looks like a single point charge from far away, the net charge is zero, so $\mathbf{E} \to 0$.)

Problem 2.3



$$E_z = \frac{1}{4\pi\epsilon_0}\int_0^L \frac{\lambda\,dx}{\mathscr{r}^2}\cos\theta;\ (\mathscr{r}^2 = z^2 + x^2;\ \cos\theta = \tfrac{z}{\mathscr{r}})$$
$$= \frac{1}{4\pi\epsilon_0}\lambda z\int_0^L \frac{1}{(z^2+x^2)^{3/2}}dx$$
$$= \frac{1}{4\pi\epsilon_0}\lambda z\left[\frac{1}{z^2}\frac{x}{\sqrt{z^2+x^2}}\right]\Bigg|_0^L = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{z}\frac{L}{\sqrt{z^2+L^2}}.$$
$$E_x = -\frac{1}{4\pi\epsilon_0}\int_0^L \frac{\lambda\,dx}{\mathscr{r}^2}\sin\theta = -\frac{1}{4\pi\epsilon_0}\lambda\int\frac{x\,dx}{(x^2+z^2)^{3/2}}$$
$$= -\frac{1}{4\pi\epsilon_0}\lambda\left[-\frac{1}{\sqrt{x^2+z^2}}\right]\Bigg|_0^L = -\frac{1}{4\pi\epsilon_0}\lambda\left[\frac{1}{z} - \frac{1}{\sqrt{z^2+L^2}}\right].$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{z}\left[\left(-1 + \frac{z}{\sqrt{z^2+L^2}}\right)\hat{\mathbf{x}} + \left(\frac{L}{\sqrt{z^2+L^2}}\right)\hat{\mathbf{z}}\right].}$$

For $z \gg L$ you expect it to look like a point charge $q = \lambda L$: $\mathbf{E} \to \frac{1}{4\pi\epsilon_0}\frac{\lambda L}{z^2}\hat{\mathbf{z}}$. It checks, for with $z \gg L$ the $\hat{\mathbf{x}}$ term $\to 0$, and the $\hat{\mathbf{z}}$ term $\to \frac{1}{4\pi\epsilon_0}\frac{\lambda}{z}\frac{L}{z}\hat{\mathbf{z}}$.

Problem 2.4

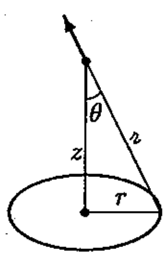
From Ex. 2.1, with $L \to \frac{a}{2}$ and $z \to \sqrt{z^2 + \left(\frac{a}{2}\right)^2}$ (distance from center of edge to P), field of *one* edge is:

$$E_1 = \frac{1}{4\pi\epsilon_0}\frac{\lambda a}{\sqrt{z^2 + \frac{a^2}{4}}\sqrt{z^2 + \frac{a^2}{4} + \frac{a^2}{4}}}.$$

There are 4 sides, and we want vertical components only, so multiply by $4\cos\theta = 4\frac{z}{\sqrt{z^2+\frac{a^2}{4}}}$:

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{4\lambda a z}{\left(z^2 + \frac{a^2}{4}\right)\sqrt{z^2 + \frac{a^2}{2}}}\hat{\mathbf{z}}.}$$

Problem 2.5



"Horizontal" components cancel, leaving: $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left\{\int\frac{\lambda dl}{\mathscr{r}^2}\cos\theta\right\}\hat{\mathbf{z}}$.
Here, $\mathscr{r}^2 = r^2 + z^2$, $\cos\theta = \frac{z}{\mathscr{r}}$ (both constants), while $\int dl = 2\pi r$. So

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda(2\pi r)z}{(r^2 + z^2)^{3/2}}\hat{\mathbf{z}}.}$$

Problem 2.6

Break it into rings of radius r, and thickness dr, and use Prob. 2.5 to express the field of each ring. Total charge of a ring is $\sigma \cdot 2\pi r \cdot dr = \lambda \cdot 2\pi r$, so $\lambda = \sigma dr$ is the "line charge" of each ring.

$$E_{\text{ring}} = \frac{1}{4\pi\epsilon_0}\frac{(\sigma dr)2\pi r z}{(r^2 + z^2)^{3/2}};\quad E_{\text{disk}} = \frac{1}{4\pi\epsilon_0}2\pi\sigma z\int_0^R \frac{r}{(r^2 + z^2)^{3/2}}dr.$$

$$\boxed{\mathbf{E}_{\text{disk}} = \frac{1}{4\pi\epsilon_0}2\pi\sigma z\left[\frac{1}{z} - \frac{1}{\sqrt{R^2 + z^2}}\right]\hat{\mathbf{z}}.}$$

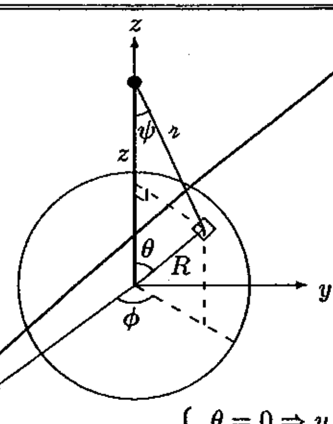
For $R \gg z$ the second term $\to 0$, so $\mathbf{E}_{\text{plane}} = \frac{1}{4\pi\epsilon_0} 2\pi\sigma\hat{\mathbf{z}} = \boxed{\frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}.}$

For $z \gg R$, $\frac{1}{\sqrt{R^2+z^2}} = \frac{1}{z}\left(1+\frac{R^2}{z^2}\right)^{-1/2} \approx \frac{1}{z}\left(1-\frac{1}{2}\frac{R^2}{z^2}\right)$, so $[\,] \approx \frac{1}{z} - \frac{1}{z} + \frac{1}{2}\frac{R^2}{z^3} = \frac{R^2}{2z^3}$,
and $E = \frac{1}{4\pi\epsilon_0}\frac{2\pi R^2\sigma}{2z^2} = \frac{1}{4\pi\epsilon_0}\frac{Q}{z^2}$, where $Q = \pi R^2\sigma$. ✓

Problem 2.7

$\mathbf{E}$ is clearly in the z direction. From the diagram,
$dq = \sigma da = \sigma R^2 \sin\theta\, d\theta\, d\phi$,
$\imath^2 = R^2 + z^2 - 2Rz\cos\theta$,
$\cos\psi = \frac{z - R\cos\theta}{\imath}$.

So



$$E_z = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma R^2 \sin\theta\, d\theta\, d\phi (z - R\cos\theta)}{(R^2+z^2-2Rz\cos\theta)^{3/2}}. \qquad \int d\phi = 2\pi.$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_0^\pi \frac{(z-R\cos\theta)\sin\theta}{(R^2+z^2-2Rz\cos\theta)^{3/2}}\,d\theta. \qquad \text{Let } u = \cos\theta;\ du = -\sin\theta\, d\theta; \left\{\begin{array}{l}\theta = 0 \Rightarrow u = +1\\ \theta = \pi \Rightarrow u = -1\end{array}\right\}.$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_{-1}^{1} \frac{z - Ru}{(R^2+z^2-2Rzu)^{3/2}}\,du. \quad \text{Integral can be done by partial fractions—or look it up.}$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\left[\frac{1}{z^2}\frac{zu - R}{\sqrt{R^2+z^2-2Rzu}}\right]_{-1}^{1} = \frac{1}{4\pi\epsilon_0}\frac{2\pi R^2\sigma}{z^2}\left\{\frac{(z-R)}{|z-R|} - \frac{(-z-R)}{|z+R|}\right\}.$$

For $z > R$ (outside the sphere), $E_z = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^2\sigma}{z^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}$, so $\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}}.}$

For $z < R$ (inside), $E_z = 0$, so $\boxed{\mathbf{E} = 0.}$

Problem 2.8

According to Prob. 2.7, all shells *interior* to the point (i.e. at smaller r) contribute as though their charge were concentrated at the center, while all exterior shells contribute nothing. Therefore:

$$\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{int}}}{r^2}\hat{\mathbf{r}},$$

where Q_{int} is the total charge interior to the point. *Outside* the sphere, *all* the charge is interior, so

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}.}$$

Inside the sphere, only that fraction of the total which is interior to the point counts:

$$Q_{\text{int}} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}Q = \frac{r^3}{R^3}Q, \quad \text{so } \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{r^3}{R^3}Q\frac{1}{r^2}\hat{\mathbf{r}} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r}.}$$

Problem 2.9

(a) $\rho = \epsilon_0 \nabla\cdot\mathbf{E} = \epsilon_0\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot kr^3\right) = \epsilon_0\frac{1}{r^2}k(5r^4) = \boxed{5\epsilon_0 kr^2.}$