Problem 1.49 (a) $\nabla\cdot\mathbf{F}_1 = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}\left(x^2\right) = \boxed{0}$; $\nabla\cdot\mathbf{F}_2 = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = \boxed{3}$

$$\nabla\times\mathbf{F}_1 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x^2 \end{vmatrix} = -\hat{\mathbf{y}}\frac{\partial}{\partial x}\left(x^2\right) = \boxed{-2x\hat{\mathbf{y}}}; \quad \nabla\times\mathbf{F}_2 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \boxed{0}$$

$\vec{A}_1 = \frac{1}{3}x^3\hat{y}$

$\boxed{\mathbf{F}_2 \text{ is a gradient; } \mathbf{F}_1 \text{ is a curl}}$ $\boxed{U_2 = \frac{1}{2}\left(x^2+y^2+z^2\right)}$ would do ($\mathbf{F}_2 = \nabla U_2$).

For $\mathbf{A}_1$, we want $\left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right) = \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) = 0$; $\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = x^2$. $A_y = \frac{x^3}{3}$, $A_x = A_z = 0$ would do it.

$\cancel{\boxed{\mathbf{A}_1 = \frac{1}{3}x^2\hat{\mathbf{y}}}}$ ($\mathbf{F}_1 = \nabla\times\mathbf{A}_1$). (But these are not unique.)

(b) $\nabla\cdot\mathbf{F}_3 = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$; $\nabla\times\mathbf{F}_3 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \hat{\mathbf{x}}(x-x) + \hat{\mathbf{y}}(y-y) + \hat{\mathbf{z}}(z-z) = 0$

So $\mathbf{F}_3$ can be written as the gradient of a scalar ($\mathbf{F}_3 = \nabla U_3$) and as the curl of a vector ($\mathbf{F}_3 = \nabla\times\mathbf{A}_3$). In fact, $\boxed{U_3 = xyz}$ does the job. For the vector potential, we have

$$\left\{\begin{array}{lll} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = yz, & \text{which suggests} & A_z = \frac{1}{4}y^2z + f(x,z);\; A_y = -\frac{1}{4}yz^2 + g(x,y) \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = xz, & \text{suggesting} & A_x = \frac{1}{4}z^2x + h(x,y);\; A_z = -\frac{1}{4}zx^2 + j(y,z) \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = xy, & \text{so} & A_y = \frac{1}{4}x^2y + k(y,z);\; A_x = -\frac{1}{4}xy^2 + l(x,y) \end{array}\right\}$$

Putting this all together: $\boxed{\mathbf{A}_3 = \frac{1}{4}\left\{x\left(z^2-y^2\right)\hat{\mathbf{x}} + y\left(x^2-z^2\right)\hat{\mathbf{y}} + z\left(y^2-x^2\right)\hat{\mathbf{z}}\right\}}$ (again, not unique).