

**Generations and properties on narrow electron flows in mesoscopic solid-state structures**

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A method for the evaluation of the angular width of an electron beam generated by a nanoconstriction is proposed and demonstrated. The approach is based on analysis of a narrow-width electron flow that quantizes into modes inside a confining constriction, which is described in the adiabatic approximation, evolving into a freely propagating electronic state after exiting the constriction. The method that we developed allows us to find the parameters and the shape of the constriction that are optimal for generation of extremely narrow electron beams. In the case of a constriction characterized by a linear widening shape, an asymptotically exact solution for the injection problem is found. That solution verifies semiquantitative results related to the angular characteristics of the beam, and it opens the way for determination of the distribution function of the electrons in the beam. We have found the relationship between the angular distribution of the electron density in the beam and the quantum states of the electrons inside the constriction. Such narrow electron beams may be employed in investigations of electronic systems and in data manipulations in electronic and spintronic devices.

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**I. INTRODUCTION**

Microconstrictions (also referred to as point contacts) connecting macroscopic reservoirs are of particular interest in efforts aimed at the generation and investigation of ballistic quasiparticle transport in solids.<sup>1</sup> Recently, the development of methods for imaging electron flows attracted significant attention<sup>2-8</sup> due to its potential to unveil the details of electron motion in low-dimensional systems and to provide insights into the behavior of devices in the quantum regime. Moreover, with the use of a most recently developed erasable electrostatic lithographic technique,<sup>9</sup> creation of quantum constrictions with desired shapes has been demonstrated. Additionally, metallic nanowires with high carrier density<sup>10</sup> may also hold some promise as devices for injection of electron flows. In light of above, the problem of determining the operational parameters of an electron beam injected through a constriction with a highly reduced size is both timely and important.

An electron flow injected through a constriction is in general anisotropic. One of the first demonstrations of the importance of the velocity anisotropy in electron flows can be found in experiments with electron beams injected by quantum point contacts,<sup>11,12</sup> where a collimation effect<sup>13</sup> was found (see also Ref. 14). The relative angular narrowness of an electron beam allows experimental determination of the electron-electron relaxation time.<sup>15-18</sup> In the scattering spectroscopy method proposed and demonstrated in Ref. 19, the narrowness of the electron beam plays a key role: that is, the ability to control the scattering angle by means of a narrow-angle beam injector, as well as a detector, allows one to determine experimentally the electronic angle-dependent differential scattering cross sections associated with different types of scatterers. Consequently, a narrow electron beam may serve as a powerful tool for studying the properties of

electron scattering processes and for determination of the characteristics of the electron gas.

Narrow electron beams may also serve as most effective tools for the transmission of information in microdevices and nanodevices (including transportation of spin-polarized states<sup>20</sup>) and as instruments for handling the spin and the charge states of quantum memory cells. In this context, we remark that issues pertaining to the angular and spatial distribution of narrow electron beams are of great significance for the development of high-resolution experimental techniques that utilize such beams, as well as for the development and application of accurate spatially targeted transfer of information using narrow electron flows. We note here that, to date, the smallest angular width of an electron beam injected into a two-dimensional electron gas (2DEG) by a quantum point contact is of the order of  $10^\circ$ ; in Ref. 19 an angular width  $\Phi \approx 12^\circ$  was observed (while Refs. 2 and 3 reported a width  $\Phi \approx 6^\circ$ , it corresponds only to the most pronounced central part of the electron flow).

The main goal of our work is to analyze issues pertaining to the prospect of generating super-narrow electron beams. To this end, we study also the distribution function of electrons in the beam, since it raises considerations related to the selection of conditions for the formation of narrow beams. The interest in conductance quantization in quantum two-dimensional and three-dimensional constrictions (such as point contacts, nanowires, and atomic chains)<sup>21-26</sup> led to intensive investigations of the electronic states in these systems. One of the main characteristics of this phenomenon relates to the fact that the quantized staircaselike variation of the conductance (with gate voltage or constriction width) is determined by the adiabatic properties of the constriction, and it is rather insensitive to details of the geometrical configuration; here, “adiabatic” means a slow dependence of the constriction width  $2r$  on the coordinate  $z$  along the longitu-

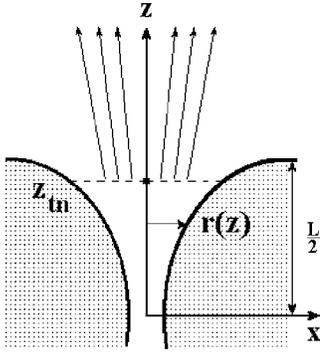


FIG. 1. Schematic of the constriction and an injected beam. The length of the constriction  $L$  is taken such that the detachment point  $z_m$  is located inside the constriction.

dinal axis of the constriction (see Fig. 1). The width changes noticeably on a scale that exceeds essentially the minimal width  $r(0)$  (see Ref. 21). However, the problem of the states of electrons that have passed through the constriction has not been solved in the general case of the adiabatic approximation, since the transformation of the adiabatic quantum states inside the constriction to the distribution of freely moving electrons occurs in a region where the adiabatic approximation ceased to be valid. Nevertheless, in Ref. 13 the characteristics of an electron beam injected by a constriction have been studied in the adiabatic approximation using the classical adiabatic invariant  $I = p_x(z)r(z)$ , where  $p_x$  is the  $x$  component of the electron momentum (see Fig. 1). Due to the conservation of the adiabatic invariant  $I$ , the beam converges (the flaring effect<sup>13</sup>) with increasing  $z$ , and near the exit of the constriction we have

$$\sin\left(\frac{\Phi}{2}\right) = \frac{r(0)}{r_{max}}, \quad (1)$$

where  $r_{max}$  is the half width of the constriction at the exit, and  $r(0)$  is the half width at  $z=0$  (the origin of the  $z$  axis is taken at the middle of the constriction). This result<sup>13</sup> is valid, as will be shown in Sec. I, only for relatively “short” constrictions where the adiabatic approximation is effectively valid for the entire constriction. A simulation of the classical trajectories of the particles in such constrictions has been presented in Ref. 12 and has been used to determine the angular width of the beam.

In Sec. I, we propose an approach that allows us to describe qualitatively the motion of electrons exiting from the adiabatic region and, thus, it permits analysis of the angular characteristics of a beam injected by a constriction of an arbitrary shape. In this case the parameters of the constriction become particularly important at distances exceeding the characteristic length scale that determines the conductance quantization behavior.

In Sec. II, we find an asymptotically exact solution for electron states in a constriction modeled by a linear widening. This solution describes the conversion of adiabatic states inside the constriction into states described by semiclassical wave functions outside it, and it supports the results of the qualitative study. The “linear” constriction that we study here

is also of additional interest since we find that in such a constriction the pattern of the distribution of the electronic density inside the constriction is maintained when the electrons move away from the exit. Such distributions were observed in Refs. 4 and 8 using scanning probe microscopy (see also Ref. 5 and references therein).

In Sec. III, we consider the electronic distribution function of the injected beam and compare our results with those of Refs. 4–8, 13, and 27. We analyze the conditions when the distribution of electrons in the beam reproduces the probability density function inside the constriction; a distribution of this type has been observed in Refs. 4 and 6–8. We find also the electron distribution in the opposite limiting case where the constriction shape varies in a less smooth manner.

For the sake of simplicity, we limit ourselves here to two-dimensional (2D) constrictions, noting that the extension of our results to the three-dimensional case is rather straightforward. Additionally, we neglect electron-impurity scattering and consider only the ballistic regime (which is readily achievable in 2D heterostructure systems, e.g., see Ref. 12). Because of the scattering of electrons by the donor atom density fluctuations (in 2D heterostructures) and by impurities,<sup>5</sup> the electron flow may form narrow branches with apparently small changes in the total angular width of the flow. An additional widening (spreading) of the electron flow may be estimated (in a diffusive approach) as  $\Delta\Phi \sim \Phi_0\sqrt{z}/z_0, z \gg z_0$  (here  $z$  is the distance along the propagation axis from the point contact,  $z_0$  is the mean scale of the spatial fluctuations of the scattering potential, and  $\Phi_0$  is an average angular deviation of the electrons due to the interactions with the fluctuations of the underlying potential). We remark that the distance dependence of the angular widening of the beam caused by electron-electron interactions (see Ref. 28) is quite different from the above expression.

## II. INJECTION CONDITIONS FOR NARROW BEAMS

Let us consider a constriction with an adiabatic narrow region; apparently, other types of constrictions have been commonly found to be unsuitable as effective injectors of narrow beams. Note that the approach of Ref. 13, which is based on employment of an adiabatic invariant, may be generalized to take into account the quantization of the energy of transverse motion (along the  $x$  direction) of electrons in the constriction.<sup>21</sup> It is known (see, for example, Refs. 29 and 30) that in the semiclassical approximation, the adiabatic invariant is quantized in units of  $\hbar$ . Qualitatively, we may write for all the electron states in the constriction

$$I = p_{xn}(z)r_n(z) \approx \hbar(n + \gamma)\beta, \quad (2)$$

where  $n=1, 2, \dots$  is a discrete quantum number,  $p_{xn}(z)$  and  $r_n(z)$  are the root-mean-square values of  $p_x$  and  $x$ , respectively, in the  $n$ th quantum state, and  $\gamma$  and  $\beta$  are numerical constants (of the order of unity) which depend on the model of the confinement potential.

Let us show that the role of the breakdown of the adiabatic approximation in the formation of a beam may be analyzed via the use of a simple picture of “detachment” of the beam from the constriction walls (at least for constrictions

where the sign of the wall curvature remains the same throughout). Detachment of the beam occurs when the opening angle of the particles in the constrictions [of the order of  $p_{zn}(z)/p_{zn}(z)$  that decreases with the distance from the center due to the increase of  $r_n(z)$ ] becomes smaller than the corner angle of the constriction  $dr_n(z)/dz$ . Thus, the “detachment point”  $z_m$  (see Fig. 1) for the  $n$ th mode of the beam may be determined from the following equations

$$p_{zn}(z)r_n(z)\frac{dr_n(z)}{dz} = \hbar(n + \gamma)\beta, \quad (3)$$

$$p_{zn}(z) = \sqrt{2m[\varepsilon_F - \varepsilon_n(z)]}. \quad (4)$$

Here,  $\varepsilon_n(z)$  and  $p_{zn}(z)$  are, respectively, the energy of transverse motion and the  $z$  component of the momentum, which are well-defined values in the adiabatic approximation,<sup>21</sup>  $m$  is the effective mass, and  $\varepsilon_F$  is the Fermi energy of the electrons in the wide region; we assume that the voltage drop across the constriction is small enough, that is  $eV \ll \varepsilon_F$ . The condition of the reality of  $p_{zn}(0)$  determines the number  $n_{max}$  associated with the last mode which can pass through the constriction. The angular width  $\Phi_n$  of the  $n$ th component of the beam is given by the following equation  $\sin(\Phi_n/2) = p_{zn}(z_m)/p_F$ , since the transverse momentum of an electron remains constant when the electron exits the constriction and  $|p| = p_F = \sqrt{2m\varepsilon_F}$  far from the constriction. Taking into account Eq. (2) we obtain

$$\sin\left(\frac{\Phi_n}{2}\right) \approx \frac{\hbar(n + \gamma)\beta}{p_F r_n(z_m)}. \quad (5)$$

This equation takes into account a possible variation of  $p_z$  due to the variation of the confinement potential  $U(x, z)$  at  $z > z_m$ .

Let us show next that the “detachment point”  $z_m$ , determined by Eqs. (3) and (4), coincides with the limit of validity of the adiabatic approximation. The wave function of an electron in the adiabatic approximation has the following form  $\psi = \eta_n(x; z)\varphi_n(z)$  (see Ref. 21), where the function  $\eta_n(x; z)$  satisfies the Schrödinger equation that is local with respect to  $z$

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x, z)\right)\eta_n = \varepsilon_n(z)\eta_n. \quad (6)$$

The function  $\varphi_n(z)$  is the wave function associated with longitudinal motion (along the axis of the constriction) in the field of the “effective potential”  $\varepsilon_n(z)$ . From examination of the terms in the complete Schrödinger equation that are maintained in comparison with those that are omitted in the adiabatic approximation [these include the terms  $\varphi\partial^2\eta/\partial z^2$  and  $(\partial\eta/\partial z)(\partial\varphi/\partial z)$ ], we obtain the following inequalities [in Eq. (7) primes denote derivatives with respect to  $z$ ]

$$nr_n'^2, r_n r_n'', \frac{r_n p_{zn} r_n'}{\hbar} \ll n. \quad (7)$$

These inequalities determine the region where the adiabatic approximation is valid. It is easy to check that the last inequality will break down first (or simultaneously with the

others) when  $z$  increases ( $z > 0$ ). To prove this, it is enough to consider the region where  $r_n(z) - r_n(0) > r_n(0)$ , because in this narrow region the validity of all these inequalities is equivalent to the initial assumption about the adiabatic constriction. If we assume that  $r_n$  increases monotonically with the increase of the  $z$  coordinate and that  $U(x, z)$  decreases monotonically [and, therefore,  $\varepsilon_n \approx p_{zn}^2/2m + U(0, z)$  decreases too], it follows from Eq. (4), that  $p_{zn} \geq p_{zn} \approx \hbar n/r_n$  for modes which move through the constriction, thus proving our conjecture. Therefore, the regions that are associated with the adiabatic approximation and with free propagation of the particles are adjacent to each other, and there is no intermediate region between them. This conclusion justifies our suggestion that the opening angle of the constriction  $\Phi = \Phi_{n_{max}}$  could be evaluated from Eqs. (3)–(5).

To end our discussion of Eq. (7), we note that the validity of the inequalities  $r'' \gg n/r \gg r'p_z/\hbar$  may be extended to the case that the profile of the constriction has a “break,” i.e., a small region with a large shape curvature. If  $r' \ll 1$  on both sides of the break it leads to only small corrections to the electron wave functions. Imperfections in the profile of the constriction (such as breaks or steps) which are small compared with the electron wave length have only a weak effect on the characteristics of the beam.

In the hard-wall model that we mainly use below,  $r_n(z)$  does not depend on  $n$  and it is equal to the half width of the constriction  $r(z)$ . Also,  $\varepsilon_n(z) = [\pi\hbar n/2r(z)]^2/2m + U(z)$ , where  $U(z)$  is the part of the potential that depends on the  $z$  coordinate,  $\gamma=0$ , and  $\beta=\pi/2$ . We analyze first the possibility of generating a narrow beam in a constriction with no potential barrier in the center, i.e.,  $U(z)=0$ . In this case,  $n_{max} \approx 2p_F r(0)/\pi\hbar$  and we obtain from Eq. (5)

$$\sin\left(\frac{\Phi}{2}\right) = \frac{r(0)}{r(z_{m_{max}})}. \quad (8)$$

Note that Eq. (8) is similar to Eq. (1) of Ref. 13, with the only distinction regarding the occurrence of  $r(z_{m_{max}})$ , instead of  $r_{max}$ . Since we consider here a narrow beam,  $\Phi \ll 1$ , in order to find the detachment point we may analyze Eq. (3) far away from the center of the constriction, where  $r(z) \gg r(0)$  and where, following Eq. (4),  $p_{zn} \approx p_F$ . Let the shape of the constriction in this region be given by the following power dependence:  $r(z) = a|z|^\alpha$  [from the evident condition  $r(z_{m_{max}}) \gg r(0)$  we readily conclude that  $a \ll (r(0))^{1-\alpha}$ ]. In this case, from Eqs. (3) and (8) and the aforementioned estimate for  $n_{max}$  and  $p_{zn}$ , we obtain expressions for the detachment point  $z_m$  and angular width  $\Phi$ . In particular, from these formulas follows the following relation:  $z_{m_{max}} \approx 4\alpha r(0)/\Phi^2$ . This relation determines the minimal length  $L$  of the constriction (see Fig. 1) which produces the electron beam with the given angular width  $\Phi$

$$L \approx z_{m_{max}}, \quad z_{m_{max}} \approx 4\alpha \frac{r(0)}{\Phi^2}. \quad (9)$$

According to Eq. (1), the reduction of the constriction length  $L$  down to the length scale less than  $z_{m_{max}}$  (when the constriction ends abruptly in the adiabatic region) leads to

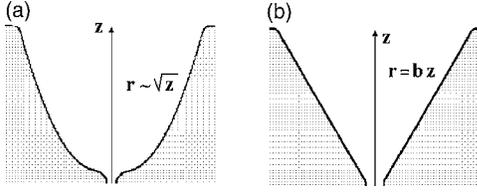


FIG. 2. Constrictions of different shapes: (a) a parabolic constriction, with  $r^2 \approx r(0)z$  at  $r \gg r(0)$ , and (b) a linear widening constriction.

increasing of the angular width of the beam. On the other hand, an increase of  $L$  beyond the length scale determined by Eq. (9) does not change  $\Phi$ , because the beam had been already detached from the constriction walls earlier on the length scale  $z_{in}$ . In other words, to generate a flow with an angular width  $\Phi$  one may need to use a constriction with an effective length that is not smaller than  $z_{in,max}$ , as determined in Eq. (9). Therefore, we conclude that the “flaring effect”<sup>13</sup> produces narrow beams only for relatively long constrictions.

Decreasing the relative length of the constriction is related to a decrease of the exponent  $\alpha$ . It is evident that the detachment of a beam is possible only if  $\alpha > 1$ . Nevertheless, if  $\frac{1}{2} < \alpha < 1$ , the condition  $z \ll z_{in,max} \approx [r(0)/a^2]^{1/(2\alpha-1)}$  determines the adiabatic region. At  $z \gg z_{in,max}$  the propagation of the electrons can be described in terms of classical mechanics. It is possible to verify that Eq. (9) remains valid in this case and that the optimal length of the constriction (required in order to generate a narrow beam) can be estimated to be of the order of  $z_{in,max}$ .

The case when  $\alpha = \frac{1}{2}$  is of special interest. When  $a^2 = 2r$  and  $p_z \approx p_F$ , Eq. (3) can be used for all values of  $z$ , and the adiabatic condition is fulfilled everywhere in the constriction. Thus, for  $\alpha = \frac{1}{2}$ , Eq. (9) is valid for any length of constriction (if  $\Phi \ll 1$ ). This differs from the case of  $\alpha > \frac{1}{2}$ , where, as aforementioned, an increase of  $L$  beyond the detachment point  $z_t$  does not reduce the angular width of the beam. When  $\Phi \ll 1$ , see Eq. (9),  $L \approx 2r(0)/\Phi^2$  [at  $a^2 \approx r(0)$ ] will be valid for arbitrary length of the constriction. In the case where  $\alpha < \frac{1}{2}$  the relation between the relative length and the angle  $\Phi$  is less favorable in the adiabatic region  $z \gg z_{in,max}$ . Therefore, a constriction of parabolic shape,  $r^2 \approx r(0)z$  (see Fig. 2), is the optimal choice. The case when  $\alpha = 1$  will be discussed in detail in the next section.

For a model of a “square” constriction<sup>21</sup>  $r = r(0) + 2z^2/R$ , with  $r(0) \ll R$ , and from Eqs. (8) and (9) we obtain for  $\Phi \ll 1$

$$\Phi \approx 4[r(0)/R]^{1/3}, \quad L \approx \frac{1}{2}[r(0)R^2]^{1/3}. \quad (10)$$

From this expression, we conclude that the distance scale for formation of an electron beam is larger than the distance [of the order of  $(r(0)R)^{1/2}$ ] that determines the conductance quantization.

The potential barrier in the center of constriction may also lead to the narrowing of the electron flow.<sup>13</sup> The cause is that in addition to the flaring effect with increasing  $z$ , the  $p_z$  com-

ponent of the momentum increases also due to the influence of the potential  $U(z)$ .

In the hard-wall approximation, we may write Eqs. (3)–(5) for  $n = n_{max}$  in the following form

$$p_F(z_t)r(z_t)\frac{dr}{dz} = p_F(0)r(0),$$

$$p_F(z) = \sqrt{2m[\varepsilon_F - U(z)]},$$

$$\Phi \approx 2\frac{p_F(0)r(0)}{p_Fr(z_t)}. \quad (11)$$

Here we assume also that  $\Phi \ll 1$  and  $p_z(z_t) \approx p_F(z_t)$ . As may be seen from Eq. (11), the flaring effect and the effect of the potential are independent from each other only when  $U(z) = \text{const}$  at  $z < z_t$ ; otherwise the potential barrier leads to a reduction of  $r(z_t)$ , i.e., it results in an attenuation of the flaring effect. Thus, in the case of a linear constriction, i.e.,  $r \propto z$ , the two effects will compensate each other [if  $U(z) = 0$  at  $z > z_t$ ]; the opening angle does not vary when the potential is switched on, but the optimal relative length,  $L \approx z_t$ , is reduced.

An alternative way to obtain a narrow beam, without having to resort to the use of a long constriction, consists of the application of an added repulsive potential. For a sufficiently wide constriction [ $r(0) \gg \lambda_F \equiv 2\pi\hbar/p_F$  and a length that exceeds slightly the width] it is sufficient to apply a potential that is transparent for one mode ( $n=1$ ) only, i.e.,  $\varepsilon_F - U(0) = [\pi\hbar/r(0)]^2/8m$ . From Eq. (1), we obtain an opening angle  $\Phi \approx \lambda_F/r(0)$  [for short constriction  $r(z_t) \approx r(0)$ ].

Note that Eqs. (8)–(11) do not include the Planck constant—indeed, they use only a classical adiabatic invariant and classical considerations pertaining to the breakdown of adiabaticity (the detachment of the beam). But, if we would like to minimize both the angular and spatial width (that is the transverse size) of the beam near the exit from the constriction we have to take into account the minimal product of these values,  $r(z_{in,max})\Phi \approx \lambda_F$ , allowed by the uncertainty principle. This underlies the finding that in order to obtain an “integrally” narrow beam, one has to use a metallic constriction with a small electron wavelength at the Fermi level. Here an “integrally” narrow beam means an electron flow with both the transverse width of the flow and the angular spreading restricted to small values.

### III. BEAM INJECTION BY A LINEAR SHAPE CONSTRICTION

Let us consider here the electron states in a constriction characterized by a linear-widening shape [see Fig. 2(b)], i.e.,  $r = bz$  at  $r \gg r(0)$ . We show below that when  $b \ll 1$  this problem has a simple, and an asymptotically exact, solution. Note that a constriction with a linear widening shape is a special case of a hyperbolic constriction. In this case the variables in the Schrödinger equation can be separated, thus allowing one to obtain a solution for the conductance in this type of contact.<sup>23</sup>

We use the aforementioned fact that  $p_x$  decreases in an adiabatic widening when the electron propagates from  $r(0)$  to  $r \gg r(0)$ . This underlies the validity of the inequalities  $p_x \ll p \approx \sqrt{2m\varepsilon}$  and  $(p - p_z) \ll p$ . The electron wave function may be written in the form

$$\Psi(x, z) = \psi(x, z) \exp\left(i \frac{pz}{\hbar}\right). \quad (12)$$

Using the hard-wall model in the linear section of the constriction and taking into account that the value of the

component  $p_z$  is close to the total momentum  $p$ , we may neglect in the Schrödinger equation the second derivative of  $\psi$  with respect to  $z$ , that is

$$\left(\frac{\hbar^2}{2m}\right) \frac{\partial^2 \psi}{\partial x^2} + i \frac{\hbar p}{m} \frac{\partial \psi}{\partial z} = 0. \quad (13)$$

It is readily observed that the solutions of Eq. (13) with a vanishing boundary condition,  $\psi[x=r(z), z]=0$ , have the following form

$$\psi_n = \begin{cases} \frac{1}{\sqrt{bz}} \sin\left[\frac{\pi n}{2} \left(\frac{x}{bz} + 1\right)\right] \exp\left\{\frac{ip}{2\hbar z} \left[x^2 + \left(\frac{\pi n \hbar}{2bp}\right)^2\right]\right\}, & x < bz, \\ 0, & x > bz. \end{cases} \quad (14)$$

Using these functions for estimations of the omitted term in the Schrödinger equation, we observe that our initial assumption is valid if  $b \ll 1$  and  $z \gg n\lambda_F/b$  [the omitted term is less than the second one on the left-hand side of Eq. (13)]. Taking into account that  $n_{max} \approx r(0)/\lambda_F$  for electron modes passing through the constriction, we find that the last inequality is equivalent to the condition  $r \gg r(0)$ .

When  $z \ll n\lambda_F/b^2$ , we can neglect the  $x^2$  dependence of the exponent in Eq. (14) compared with the  $x$  dependence of the trigonometric function and, consequently, the wave function  $\Psi_n$  has an adiabatic form.<sup>21</sup> If  $z \gg n\lambda_F/b^2$  ( $p \approx p_F = 2\pi\hbar/\lambda_F$ ) the wave function in Eq. (14) describes (in the semiclassical approximation) a beam of quasiparticles (whose distribution function we discuss in the next section), which propagates freely inside a solid angle  $\Phi = 2 \arctan(b)$ . In some sense, the detachment of the beam from the side walls occurs also in the linear constriction—here, when  $z \gg n\lambda_F/b^2$  particles “glide” along the walls and thus one can neglect their interaction with the walls. Therefore, the solution given in Eq. (14) allows us to trace the transformation of the adiabatic modes inside the constriction to the beam states described by the classical distribution function.

We remark that the asymptotically valid result (14) supports the approximate approach developed in the previous section. In the case of the linear constriction, taking into account estimates of  $n_{max}$  and  $p_z$  from Sec. II, we obtain from Eq. (3) the following result,  $z_m = r(0)n/b^2 n_{max}$ . This result coincides approximately with the limit of the adiabatic region for each of the modes,  $n\lambda_F/b^2$ . Equations (8) and (9), which follow from the approximate approach, are also valid. It is important that when  $b \ll 1$ , the limit of the adiabatic region is placed in the domain of applicability of the solution given by Eq. (14),  $r \gg r(0)$ . Thus, the solution in Eq. (14) can be matched with an adiabatic wave function<sup>21</sup> that corresponds to small  $z$ , where the shape of the constriction deviates from the linear form. Consequently, the single inequality  $b \ll 1$ , permits us to describe analytically the electron state for all values of the coordinate  $z$ .

Note also that a solution of the type given in Eq. (14) may be obtained in the “soft-wall” model for certain types of potentials forming the constriction. Let us use in the following a potential given by  $U(x, z) = z^{-2}u(x/z)$ , and let  $\eta_n$  denote the solutions of the “local” Schrödinger equation with eigenvalues  $\tilde{\varepsilon}_n$

$$\frac{\hbar^2}{2m} \eta_n'' + u \eta_n = \tilde{\varepsilon}_n \eta_n. \quad (15)$$

Here the derivatives are taken with respect to  $x/z$ . An equation similar to Eq. (13) is obtained by

$$\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} - U(x, z) \psi + \frac{i\hbar p}{m} \frac{\partial \psi}{\partial z} = 0. \quad (16)$$

The solutions of Eq. (16) are

$$\psi_n = \frac{1}{\sqrt{z}} \eta_n\left(\frac{x}{z}\right) \exp\left(\frac{i}{\hbar z} \left[\frac{px^2}{2} + \frac{\tilde{\varepsilon}_n m}{p}\right]\right). \quad (17)$$

For Eq. (16) to serve as a good approximation to the complete Schrödinger equation, the conditions  $z \gg \sqrt{\tilde{\varepsilon}_n m}/p^2$  and  $b_n \ll 1$  have to be fulfilled. An example where these conditions are fulfilled is provided by the potential  $U(x, z) = c(x^2/z^4) + d/z^2$ , where  $c$  and  $d$  are constants, and  $c \gg (\hbar n)^2/m$ . In the above,  $b_n$  may be termed as the “localization radius” of the function  $\eta_n$ . For the soft-wall potential discussed here,  $b_n$  plays (for the  $n$ th mode) the same role as the parameter  $b$  introduced earlier in the context of the hard-wall model [see the beginning of this section, Eq. (14)]; physically,  $b_n$  is the turning point in Eq. (15), corresponding to the location where  $u(x/z) = \tilde{\varepsilon}_n$  and consequently the kinetic energy vanishes there. We thus conclude that while for  $x/z < b_n$  the function  $\eta_n$  takes finite values, it decreases (typically exponentially) for  $x/z > b_n$ .

The solution given in Eq. (17) also supports our results that have been obtained in the frame of the approximate approach of Sec. II. Let us discuss the aforementioned model

of the harmonic transverse potential (at  $d=0$ ). In contrast to the hard-wall model, the parameters  $b_n$  and, consequently,  $r_n$  depend on the index  $n$ :  $r_n \propto b_n \propto (n + \gamma)^{1/2}$  (for the harmonic potential  $\gamma = -\frac{1}{2}$ ,  $n=1, 2, \dots$ ). In this model, we obtain from Eq. (3) the length of the adiabatic region which does not depend on the index  $n$ :  $z_m = r(0)/b_{n_{max}}^2 \approx \lambda_F/b_1$ , the same estimate we obtain when compare the result in (17) with the electron wave function in the adiabatic approximation.<sup>21</sup> Equations (8) and (9) are also valid for this model.

#### IV. THE DISTRIBUTION FUNCTION OF ELECTRONS IN A BEAM

The wavelength of the electron in the  $x$  direction,  $h/p_x$ , becomes less than the transverse size of the beam at a distance (along the constriction axis)  $z \gg z_{m_{max}}$  from the center of the constriction. This means that the electron wave function at this distance corresponds to the quasiclassical approximation (e.g., see Ref. 29) and wave functions (14) and (17) have the quasiclassical form  $\psi = g \exp(iS/\hbar)$ , where  $g$  is a smooth function of the  $x$  coordinate. The transverse momentum  $p_x$  and the action  $S$  are related by  $p_x = \partial S / \partial x$ . For such circumstances, the electron beam may be considered as a classical object, and the distribution function of such a classical beam, radiated from a small region, may be written as

$$f(p_x, x, z) = \rho(x, z) \delta(p_x - xp/z),$$

$$\rho(x, z) = z^{-1} \chi(x/z), \quad (18)$$

where  $\rho(x, z)$  is the distribution of the electrons with coordinates  $x$  and  $z$ . This form of the distribution function follows from a very simple fact. Namely, regardless of the type of distribution in the small region of injection, after a lapse of time (when the electrons move off from the beam axis to a distance which exceeds the initial width of the beam) the  $x$  component of the velocity of an electron,  $v_x$ , and its  $x$  coordinate will be related by  $v_x = x/t$ . Taking into account that in the case of narrow beam  $t = z/v$  with a good accuracy, we obtain Eq. (18). We suppose also that all the electrons in the beam have a definite energy,  $p^2/2m$ . The function  $\chi(\theta)$  is the angular distribution of particles, expressing the deviation from the beam axis. The distribution in Eq. (18) satisfies the condition of conservation of the particle flow, i.e.,  $\int \rho(x, z) dx = \text{const}$ .

When  $z \gg z_{m_{max}}$  the exact solution given by Eq. (14) is the semiclassical wave function (the rapid  $x$  dependence is due to the  $x^2$  term in the exponent,  $p_x = xp/z$  and it leads to the distribution function described by Eq. (18). The contribution of the  $n$ th mode to the distribution function  $\chi(\theta)$  (normalized to unity, i.e.,  $\int \chi_n(\theta) d\theta = 1$ ) has the form

$$\chi_n(\theta) = z |\Psi_n(x, z)|^2, \quad \theta = x/z, \quad (19)$$

$$|\Psi_n(x, z)|^2 = \begin{cases} (bz)^{-1} \sin^2 \left[ \frac{\pi n}{2} \left( \frac{\theta}{b} + 1 \right) \right], & |\theta| < b, \\ 0, & |\theta| > b. \end{cases} \quad (20)$$

Thus, in the linear constriction model, the density of particles in the beam reproduces exactly the contribution of the corresponding adiabatic mode to the density of particles. This is true also in the case of a constriction modeled by ‘soft’ walls [ $\chi_n = |\eta_n|^2$ , see Eq. (17)].

The above demonstrates that the linear constriction model yields an optimally smooth matching between the adiabatic states and the classical ones when the pattern of the distribution of the electronic density inside the constriction,  $|\Psi_n(x)|^2$ , is maintained as the electrons move away from the exit.

Let us consider now a constriction model that describes the opposite limit to the linear constriction discussed above—that is, when the constriction ends abruptly in the adiabatic region (this problem has been considered numerically in Ref. 27). Note first, that Eq. (13) is equivalent to the one-dimensional time-dependent Schrödinger equation; the time of motion along the  $z$  axis is  $t = zm/p$ . Consequently, when  $\Phi \ll 1$ , the problem concerning the behavior of particles leaving the adiabatic constriction can be mapped onto the one concerning determination of the response of particles initially localized in a potential well to the sudden removal of the well. The latter problem has an evident solution—i.e., in the (momentum)  $p_x$  representation, the density  $|\Psi_n(p_x)|^2$  (instead of  $|\Psi_n(x/z)|^2$ , as was the case for the linear constriction) is conserved in time. Taking into account Eq. (18) we obtain

$$\chi_n(q) = 2\pi p \hbar |\Psi_n(p_x = p\theta)|^2. \quad (21)$$

In the hard-wall potential model

$$|\Psi_n(p\theta)|^2 = \frac{n^2 r_t \sin^2 \left( kr_t + \frac{\pi n}{2} \right)}{4 \left[ (kr_t)^2 - \left( \frac{\pi n}{2} \right)^2 \right]^2 \hbar^2}, \quad k = \frac{\theta p}{\hbar}, \quad (22)$$

where  $2r_t$  is the width of the constriction at the place where the constriction terminates. The main difference between the distributions given in Eqs. (19) and (20) and Eqs. (21) and (22) is that in the first case, the distributions have the same angular size  $\Phi$  for all  $n$ , while in the second case, the distributions are localized near the angles  $\theta = \pm \pi \hbar (n-1) / 2r_t p$  (the width of the main peaks is of the order of  $\hbar / r_t p$ ).

The function described in Eq. (22) is valid for an arbitrary shape of the constriction, if we interpret  $\Psi_n(p_x)$  as the wave function of the electron at the exit of the constriction ( $(z = z_t, p_x \ll p_z)$ ). While in general this wave function differs from the one at the center of the constriction, the two are similar when the electron does not undergo any collisions with the walls after it leaves the adiabatic region. The latter takes place when the radius of curvature of the constriction in the ‘‘detachment’’ region satisfies the condition  $R \ll r_t / \Phi^2$ —this inequality is the applicability condition of Eq. (21). In the opposite limiting case, i.e., for  $R \gg r_t / \Phi^2$ , Eq. (19) is valid. Here the radius of the constriction at the detachment point  $r_t$  (where the adiabatic approximation is violated) can be determined as the maximum value of  $r$  in the region where  $dr/dz \approx \Phi$ ;  $R$  is the radius of curvature of the constriction in this region.

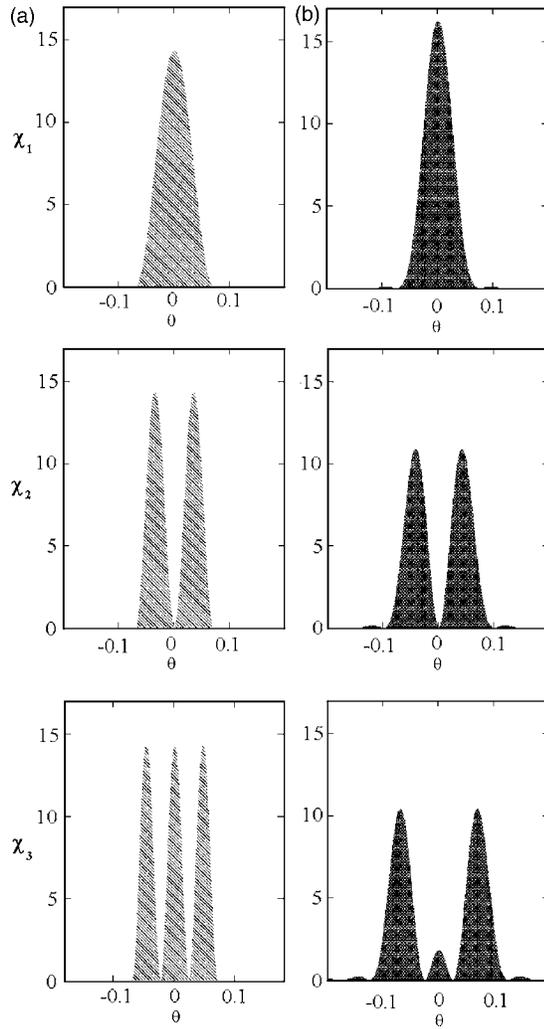


FIG. 3. The angular ( $\theta$  in radians) distribution  $\chi_n$  of the  $n$ th mode for  $n=1, 2$ , and  $3$ , plotted for (a) a constriction with a shape close to that with a linear widening, and (b) a constriction that ends abruptly ( $r_t/\lambda_F=10$ ).

The  $\theta$  dependencies of  $\chi_n$  for the first three quantum modes in the hard-wall constriction model are displayed in Fig. 3. The electron modes radiated by a constriction with a shape close to the linear widening one (a radius of curvature  $R \gg r_t/\Phi^2$ ) are displayed in Fig. 3(a). These modes reproduce the  $x$  dependence of the  $|\psi|^2$  function inside the constriction. In Fig. 3(b), we display the radiation from a constriction which ends abruptly in the adiabatic region,  $R \ll r_t/\Phi^2$ . The difference between the characteristics of the electron flows generated by the two types of constrictions is evident (compare, in particular, the angular distributions for the third mode). Note that in the model of a harmonic transverse potential (soft-wall constriction model) the distributions are the same for both types of constrictions. In this case, the wave functions are the same in the coordinate and momentum representations. We may define the angular width of the electron beam by introducing the number of modes passing through the constriction,  $n_{max}$ , and the maximal value of the  $x$  component of the electron momentum at the detachment point,  $\Phi \approx p_{xt}/p_F \sqrt{2(0.5+n_{max})}$ . These values correspond to a defi-

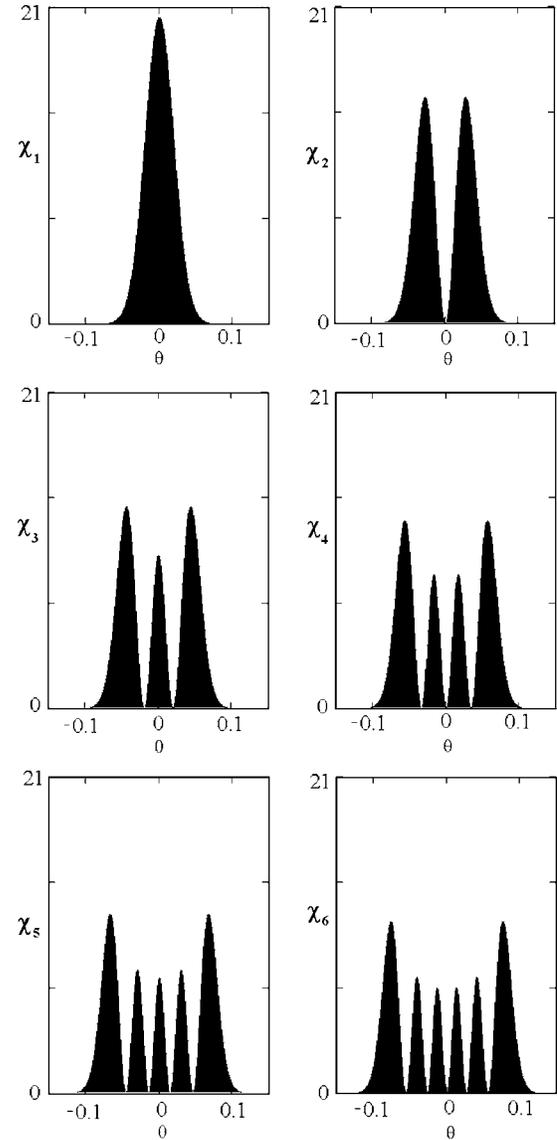


FIG. 4. The angular ( $\theta$  in radians) distribution  $\chi_n$  of the  $n$ th mode for  $n=1, 2, 3, 4, 5$ , and  $6$ , generated by a soft-wall constriction.

nite value of the coefficient in the transverse potential at the detachment point:  $c(z_t) = p_{xt}^4/8m\hbar^2(0.5+n_{max})^2$ . The corresponding half width of the electron state in the detachment point is  $r_t = 2\hbar(0.5+n_{max})/p_{xt}$ . The  $\theta$  dependencies of  $c_n$  for the first six quantum modes in the soft-wall model are displayed in Fig. 4, where we have taken the same angular width  $p_{xt}/2p_F\sqrt{2(0.5+n_{max})}$  as in Fig. 3. Similar  $\theta$  dependencies for the first three quantum modes in the harmonic confinement potential model have been discussed and observed experimentally in Refs. 3 and 4. The half width of the constriction at the detachment point satisfies the equation  $c(z_t)r_t^2 = p_{xt}^2/2m$  (this differs from the equation  $c(0)r^2 = \epsilon_F$  used widely for the definition of the width of the constriction in the narrowest region in soft-potential models).

Let us finally discuss the total electron flow injected by the constriction. This flow is a sum over all the modes that pass through the constriction

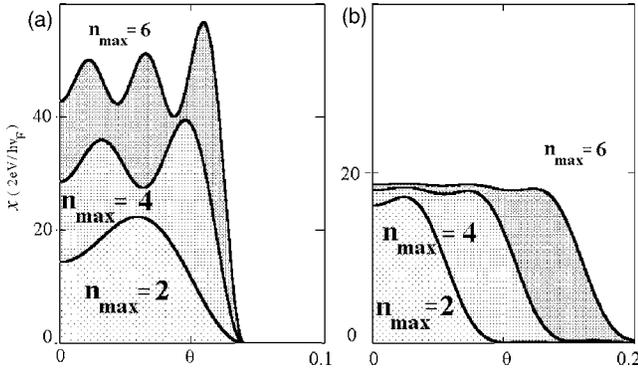


FIG. 5. The angular ( $\theta$  in radians) dependence of the electron flow (sum over all conducting modes) from a constriction, corresponding to  $n_{\max}=2,4,6$ . Results are shown for (a) a constriction with a shape close to a linear widening one, and (b) a constriction that ends abruptly. The parameters of the constrictions are as in Fig. 3.

$$\chi = \frac{VmG_0}{eP_F} \sum_{n=1}^{n_{\max}} \chi_n, \quad (23)$$

where  $V$  is the potential difference between the two reservoirs which are connected by the constriction, and  $G_0 = 2e^2/h$  is the conductance quantum. The coefficient in front of the summation is chosen in order to maintain a well-known quantization rule for the regime that is linear in  $V$  (see Ref. 26). For a sufficiently wide constriction,  $r(0) \gg \lambda_F$  and  $n_{\max} \gg 1$ , the size quantization is insignificant and this case corresponds to the classical mechanics approach. From Eqs. (19)–(23) we obtain

$$\chi = \begin{cases} \frac{2mer(0)V}{\pi^2 \hbar^2 \Phi}, & |\theta| < \Phi/2, \\ 0, & |\theta| > \Phi/2. \end{cases} \quad (24)$$

We observe that if  $n_{\max}$  is not too large, the electron beam distribution  $\chi$  oscillates with a period  $\Phi/n_{\max}$  and the amplitude of the oscillation grows at the edges of the flow at  $\theta = \pm \Phi/2$  [see Fig. 5(a),  $n_{\max}=2,4,6$ ]. The summation of the contributions of different modes radiated by the constriction which ends abruptly [Eqs. (21) and (22)] gives a result similar to Eq. (24) with additional numerically small oscillations [see Fig. 5(b)]. A  $\theta$  dependence of the beam distribution that is similar to Eq. (24) has been predicted in Ref. 13. Note that the “steplike” dependence, with sharp edges at  $\theta = \pm \Phi/2$ , is not universal. It takes place only in the classical limit for both types of constrictions discussed above. In Fig. 6, we present also the  $\theta$  dependence of the beam distribution for the soft-wall model corresponding to different values of  $n_{\max}$ . Apparently, in the classical limit, the angular distribution of the radiated electron beam that is generated by a constriction with a shape described by the expression  $r \propto z^\alpha$  for  $\alpha > 1$  (at least up to the detachment point) has no sharp edges at  $\theta = \pm \Phi/2$ .

## V. CONCLUSION

The analysis that we performed demonstrates that extremely narrow electron beams may be generated by a volt-

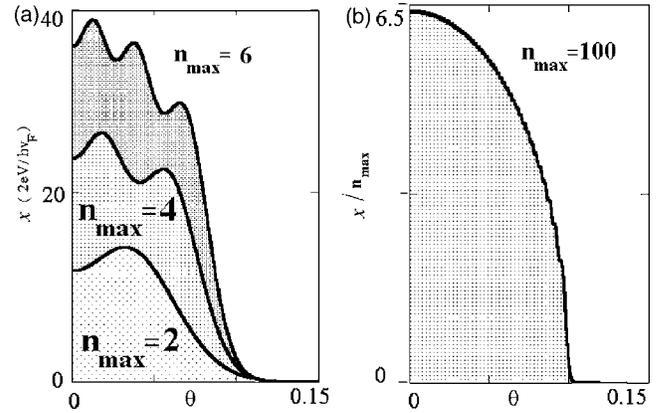


FIG. 6. The angular ( $\theta$  in radians) dependence of the electron flow (sum over all conducting modes) from a constriction. Results are shown for (a)  $n_{\max}=2,4,6$  and (b)  $n_{\max}=100$  (normalized), for a soft-wall model. The parameters of the constriction are as in Fig. 3.

age applied to sufficiently long and narrow constrictions. The minimal length  $L$  of such a constriction is related to the minimal half width  $r(0)$  and the angular size of the beam  $\Phi$  through Eq. (9).

An alternative scheme for the generation of a super-narrow electron beam may be achieved by a specially tuned electrostatic potential applied to a sufficiently wide constriction in juxtaposition with blocking of all the electronic size quantization modes in the constriction, except for the lowest one (here, the minimal width of constriction has to be much larger than the electron wavelength). To minimize the “integral” width of the beam, which combines its angular and spatial widths, one should use constrictions made of conducting materials with high electron densities.

We have also illustrated here that the angular distribution of the electron density in the beam provides information about the quantum adiabatic electronic states inside the constriction. When the adiabatic region ends smoothly, the electron density in the beam reproduces the probability density in the coordinate representation. This result elucidates the feasibility condition for the electron flow distributions observed in Refs. 3–6 and 8—accordingly, the radius of curvature of the constriction in the detachment point should be larger than  $r_t/\Phi^2$ . If the adiabatic region ends abruptly, the electron density in the beam reproduces the probability density in the momentum representation.

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