Homework #8

Probllem ~~1~~

Solution:

(a) For a set of three positive integers $(n_1, n_2, n_3)$, the electromagnetic field at thermal equilibrium in the cavity has two modes of oscillation with the frequency $\nu(n_1, n_2, n_3) = \frac{c}{2L}(n_1^2 + n_2^2 + n_3^2)^{1/2}$. Therefore, the number of modes within the frequency interval $\Delta\nu$ is

$$\left(\frac{4\pi}{8}\nu^2 \Delta\nu\right)\left(\frac{2L}{c}\right)^2 \cdot 2 .$$

Equipartition of energy then gives an energy density

$$\begin{aligned} u_\nu &= \frac{1}{L^3}\frac{dE}{d\nu} = \frac{1}{L^3} \cdot \frac{kT \cdot \frac{4\pi}{8}\nu^2 \Delta\nu \cdot \left(\frac{2L}{c}\right)^2 \cdot 2}{\Delta\nu} \\ &= 8\pi\nu^2 kT/c^3 . \end{aligned}$$

When $\nu$ is very large, this expression does not agree with experimental observations since it implies $u_\nu \propto \nu^2$.

(b) For oscillations of freqeuncy $\nu$, the average energy is

$$\begin{aligned} \frac{\sum_{n=0}^{\infty} nh\nu e^{-nh\nu/kT}}{\sum_{n=0}^{\infty} e^{-nh\nu/kT}} &= -\frac{\partial}{\partial\beta}\ln\sum_{n=0}^{\infty} e^{-\beta nh\nu}\Big|_{\beta=\frac{1}{kT}} \\ &= h\nu e^{-h\nu\beta}/(1 - e^{-h\nu\beta}) = \frac{h\nu}{e^{h\nu\beta} - 1} , \end{aligned}$$

which is to replace the classical quantity $kT$ to give

$$u_\nu = \frac{8\pi\nu^2}{c^3} \cdot \frac{h\nu}{e^{h\nu\beta} - 1} .$$



(c) The energy radiated from the hole per unit time is

$$u \propto \int_0^\infty u_\nu d\nu \propto T^4 .$$

# ~~Solution to Assignment #6~~

~~Qpcrfnn !$ 2~~ problem 2

(a) The Bose distribution is given by

$$n(k) = 1/[\exp(\beta\varepsilon(k)) - 1] \, .$$

The total number of photons is then

$$N = 2 \cdot V \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta hck/2\pi} - 1} \, ,$$

where $\varepsilon(k) = \hbar kc$ for photons and $\beta = \frac{1}{k_B T}$. The factor 2 is due to the two directions of polarization. Thus

$$n = \frac{N}{V} = \frac{1}{\pi^2} \left( \frac{k_B T}{\hbar c} \right)^3 \cdot I \, ,$$

where

$$\begin{aligned} I &= \int_0^\infty dx \frac{x^2}{e^x - 1} \\ &= \sum_{n=1}^\infty \int_0^\infty dx \cdot x^2 e^{-nx} = 2 \sum_{n=1}^\infty \frac{1}{n^3} \approx 2.4 \, . \end{aligned}$$

## Problem #~~4~~ 3

We have,

$$\epsilon = pc = \frac{hc}{\lambda_n} = \frac{nhc}{2L}$$

Now for the system to hold $N(\epsilon)$ electrons the orbitals must be filled upto $n$ (which is the radius of the sphere in the space of integers $n_x, n_y, n_z$), determined by

$$N(\epsilon) = 2 \times \frac{1}{8} \times \frac{4\pi}{3} n^3 = \frac{\pi}{3} n^3$$

We now substitute $n$ in terms of $\epsilon$ and obtain $N(\epsilon) \sim \epsilon^3$ which implies $\frac{dN}{d\epsilon} \sim \epsilon^2$.

Problem 4

Solution:

(a) The energy of black body radiation is

$$E = 2\iint \frac{d^n p d^n q}{(2\pi\hbar)^n} \frac{\hbar\omega}{e^{\hbar\omega/2\pi kT} - 1}$$
$$= \frac{2V}{(2\pi\hbar)^n} \int d^n p \frac{\hbar\omega}{e^{\hbar\omega/2\pi kT} - 1} .$$

For the radiation we have $p = \hbar\omega/c$, so

$$\frac{E}{V} = 2\left(\frac{k}{2\pi\hbar c}\right)^n k \int d^n x \frac{x}{e^x - 1} \cdot T^{n+1} ,$$

where $x = \hbar\omega/kT$. Hence $\alpha = n + 1$.

(b) The Debye Model regards solid as an isotropic continuous medium with partition function

$$Z(T,V) = \exp\left[-\hbar\sum_{i=1}^{nN}\omega_i/2kT\right]\prod_{j=1}^{nN}[1 - \exp(-\hbar\omega_j/kT)]^{-1} .$$

The Holmholtz free energy is

$$F = -kT\ln Z = \frac{\hbar}{2}\sum_{i=1}^{nN}\omega_i + kT\sum_{i=1}^{nN}\ln[1 - \exp(-\hbar\omega_i/kT)] .$$

When $N$ is very large,

$$\sum_{i=1}^{nN} \to \frac{n^2 N}{\omega_D^n}\int_0^{\omega_D}\omega^{n-1}d\omega ,$$

where $\omega_D$ is the Debye frequency. So we have

$$F = \frac{n^2 N}{2(n+1)}\hbar\omega_D + (kT)^{n+1}\frac{n^2 N}{(\hbar\omega_D)^n}\int_0^{x_D} x^{n-1}\ln[1 - \exp(-x)]dx ,$$

where $x_D = \hbar\omega_D/kT$. Hence

$$c_v = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_V \propto T^n ,$$

i.e., $\beta = n$.



(c) The theorem of equipartition of energy gives the const specific heat of a molecule as $c_v = \frac{l}{2}k$ where $l$ is the number of freedom of the molecule. For a monatomic molecule in a dimensions, $l = n$. With $c_p = c_v + k$, we get

$$\gamma = \frac{c_p}{c_v} = \frac{(n+2)}{n} .$$

Problem #5

$$N = g \sum_{\vec{k}} n_{\vec{k}} = \frac{gL}{2\pi} \int dk \quad n_{\vec{k}}$$

$$n_{\vec{k}} = \begin{cases} 1 & \text{if } |\vec{k}| < k_F \\ 0 & \text{if } |\vec{k}| > k_F \end{cases}$$

$$N = \frac{gL}{2\pi} \int_{-k_F}^{k_F} dk = \frac{g L k_F}{\pi}$$

$$k_F = \frac{\pi}{g} \frac{N}{L} = \frac{\pi}{2d} \qquad d = \frac{N}{L} = \text{atom spacing}$$

$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2 \pi^2}{8 m d^2} \simeq 2.5 \times 10^{-19} \text{ J}.$$

Problem #6 (a)

**~~Problem 3 solution:~~**

a) The grand partition function is

$$Z = \sum_{\{n_{\vec{k}}\}} \exp\left[-\beta \sum_{\vec{k}} \left[\varepsilon(\vec{k}) - \mu\right] n_{\vec{k}}\right] = \prod_{\vec{k}} \sum_{n_{\vec{k}}} \exp[-\beta[\varepsilon(k) + \mu] n_{\vec{k}}]$$

For fermions, $n_k = 0$ or 1

$$Z = \prod_{\vec{k}} \left\{1 + \exp\left[\beta\mu - \beta\varepsilon(\vec{k})\right]\right\}$$

The grand thermodynamic potential is:

$$\Omega = -k_B T \ln Z = -k_B T \sum_{\vec{k}} \ln\left[1 - \exp\{\beta\mu - \beta\varepsilon(k)\}\right]$$

$$= -k_B T V \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3} \ln\left[1 - \exp\{\beta\mu - \beta c k^\alpha\}\right]$$

$$= -\frac{V}{6\pi^2} \int_0^\infty d(k^3) \ln\left[1 - \exp\{\beta\mu - \beta c k^\alpha\}\right]$$

$$= -\frac{k_B T V}{6\pi^2} \left\{k^3 \ln\left[1 - \exp\{\beta\mu - \beta c k^\alpha\}\right]\right\}\Big|_0^\infty + \frac{k_B T V}{6\pi^2} \int_0^\infty \frac{-\exp\left(\beta\mu - \beta c k^\alpha\right)\left(-\beta\alpha c k^{\alpha-1}\right)}{1 - \exp\{\beta\mu - \beta c k^\alpha\}} k^3 dk$$

$$= \frac{k_B T V}{6\pi^2} \int_0^\infty \frac{k^3}{z^{-1}\exp(\beta c k^\alpha) - 1} d\left(\beta c k^\alpha\right)$$

$$= \frac{V}{6\pi^2 \beta (\beta c)^{3/\alpha}} (3/\alpha)! f^1_{1+3/\alpha}(z)$$

The density is:

$$n = \frac{N}{V} = -\frac{1}{V}\frac{\partial\Omega}{\partial\mu} = \frac{1}{V}\sum_{\vec{k}} \frac{1}{z^{-1}\exp[\beta\varepsilon(k)] - 1} = \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{z^{-1}\exp[\beta c k^\alpha] - 1}$$

$$= \frac{1}{2\pi^2 \alpha (\beta c)^{3/\alpha}} (3/a - 1)! f^1_{3/\alpha}(z)$$

b) The pressure is $P = \Omega/V$. The energy is

$$E = \sum_{\vec{k}} c k^\alpha n_{\vec{k}} = \sum_{\vec{k}} c k^\alpha \frac{1}{z^{-1}\exp(\beta c k^\alpha) - \eta}$$

$$= V \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3} \frac{c k^\alpha}{z^{-1}\exp(\beta p^\alpha / \hbar^\alpha) - \eta}$$

$$= \frac{V}{2\pi^2 \beta (\beta c)^{3/\alpha} \alpha} (3/\alpha)! f^1_{1+3/\alpha}(z)$$

Thus, $PV/E = \alpha/3$

# Problem #6 (b)

~~Problem 4 solution~~:

c) The grand partition function is

$$Z = \sum_{\{n_{\vec k}\}} \exp\left[-\beta \sum_{\vec k}\left[\varepsilon(\vec k)-\mu\right] n_{\vec k}\right] = \prod_{\vec k}\sum_{n_{\vec k}} \exp\left[-\beta[\varepsilon(k)+\mu] n_{\vec k}\right]$$

For bosons, $n_k = 0, 1, 2, \ldots$.

$$Z = \prod_{\vec k} \frac{1}{1-\exp\left[\beta\mu-\beta\varepsilon(\vec k)\right]}$$

The grand thermodynamic potential is:

$$\Omega = -k_B T \ln Z = k_B T \sum_{\vec k} \ln\left[1+\exp\{\beta\mu-\beta\varepsilon(k)\}\right]$$

$$= k_B T V \int_0^\infty \frac{4\pi k^2\,dk}{(2\pi)^3} \ln\left[1+\exp\{\beta\mu-\beta c k^\alpha\}\right]$$

$$= \frac{V}{6\pi^2}\int_0^\infty d(k^3) \ln\left[1+\exp\{\beta\mu-\beta c k^\alpha\}\right]$$

$$= \frac{k_B T V}{6\pi^2}\left\{k^3 \ln\left[1+\exp\{\beta\mu-\beta c k^\alpha\}\right]\right\}\Big|_0^\infty - \frac{k_B T V}{6\pi^2}\int_0^\infty \frac{\exp\left(\beta\mu-\beta c k^\alpha\right)\left(-\beta\alpha c k^{\alpha-1}\right)}{1+\exp\{\beta\mu-\beta c k^\alpha\}} k^3\,dk$$

$$= \frac{k_B T V}{6\pi^2}\int_0^\infty \frac{k^3}{z^{-1}\exp(\beta c k^\alpha)+1} d\left(\beta c k^\alpha\right)$$

$$= \frac{V}{6\pi^2 \beta(\beta c)^{3/\alpha}}(3/\alpha)!\, f^{-1}_{1+3/\alpha}(z)$$

The density is:

$$n = \frac{N}{V} = -\frac{1}{V}\frac{\partial\Omega}{\partial\mu} = \frac{1}{V}\sum_{\vec k}\frac{1}{z^{-1}\exp[\beta\varepsilon(k)]+1} = \int_0^\infty \frac{4\pi k^2\,dk}{(2\pi)^3}\frac{1}{z^{-1}\exp[\beta c k^s]-\eta}$$

$$= \frac{1}{2\pi^2 s(\beta c)^{3/s}}(3/s-1)!\, f^{\eta}_{3/s}(z)$$

d) The pressure is $P = \Omega/V$. The energy is

$$E = \sum_{\vec k} c k^\alpha n_{\vec k} = \sum_{\vec k} c k^\alpha \frac{1}{z^{-1}\exp(\beta c k^\alpha)+1}$$

$$= V\int_0^\infty \frac{4\pi k^2\,dk}{(2\pi)^3}\frac{c k^\alpha}{z^{-1}\exp(\beta p^\alpha/\hbar^\alpha)+1}$$

$$= \frac{V}{2\pi^2 \beta(\beta c)^{3/\alpha}\alpha}(3/\alpha)!\, f^{-1}_{1+3/\alpha}(z)$$

Thus, $PV/E = \alpha/3$