$$\boxed{\text{Ideal Fermi gas}}$$

For $T \to 0$ $\quad \langle n_k \rangle = \dfrac{1}{e^{\beta(\varepsilon_k - \mu)} + 1} = \begin{cases} 1 & \text{if } \varepsilon_k < \mu \\ 0 & \text{if } \varepsilon_k > \mu \end{cases}$

this limiting value $\mu(T \to 0)$ is called the Fermi energy ε_F. All one-particle states with $\varepsilon_k < \varepsilon_F$ is occupied at $T=0$. For ideal gas $\varepsilon_k = \hbar^2 k^2 / 2m$

$\Rightarrow$ There's a corresponding Fermi wave vector k_F.

Since
$$N = g\sum_{\vec{k}} \langle n_{\vec{k}} \rangle = gV \int_0^{k_F} \frac{d^3k}{(2\pi)^3} = \frac{gVk_F^3}{6\pi^2}$$

$$\Rightarrow \quad k_F = \left(\frac{6\pi^2 n}{g}\right)^{1/3} \qquad \varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3}$$

$n = N/V$

(Note classical gas has large # of microstate at $T=0$, $\Omega = V^N/N!$, but Fermi gas has only one unique ground state, $\Omega = 1$)

<u>Low temperature expansion:</u> $k_B T \ll \varepsilon_F$ $\quad (|\ln z| = |\beta\mu| \gg 1)$

$$n = g\int_0^\infty \frac{d^3k}{(2\pi)^3} \frac{1}{z^{-1}\exp\left(\beta\frac{\hbar^2k^2}{2m}\right)+1}$$

$$= \frac{g}{\lambda^3}\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{dx \cdot x^{1/2}}{z^{-1}\exp(x)+1}$$

$$\simeq \frac{g}{\lambda^3}\frac{(\ln z)^{3/2}}{(3/2)!}\left[1+\frac{\pi^2}{6}\frac{3}{4}(\ln z)^{-2}+\dots\right]$$

Solve for $\ln z$ gives

$$\ln z \simeq \beta\varepsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right]$$

Similarly the pressure

$$P \simeq P_F\left[1+\frac{5}{12}\pi^2\left(\frac{kT}{\varepsilon_F}\right)^2\right] \qquad P_F = \frac{2}{5}n\varepsilon_F$$

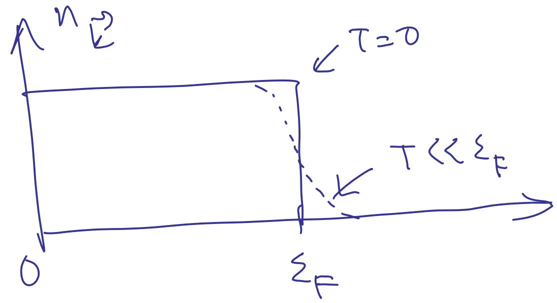
and energy $$\boxed{E = \frac{3}{2} P V = \frac{3}{2} P_F V \left[1 + \frac{5}{12}\pi^2 \left(\frac{kT}{\varepsilon_F}\right)^2 \right]}$$

Specific heat $$C_V = \left.\frac{\partial E}{\partial T}\right|_V \simeq \frac{\pi^2}{2} N k_B \frac{kT}{\varepsilon_F} \sim T$$

* μ (or $\ln z$) goes from positive ε_F at $T \to 0$ to negative value at large T.
* $C_V \sim T$ at low T because only a small # of particles $\left(\sim N \frac{kT}{\varepsilon_F}\right)$ are excited. each can absorb energy kT, so

$$\Delta E \sim kT \cdot N \frac{kT}{\varepsilon_F}$$

$$\Rightarrow C = \frac{\Delta E}{T} \simeq Nk \cdot \frac{kT}{\varepsilon_F}$$



Ideal Bose gas

$$\langle n_k \rangle_T = \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1}$$

* $\langle n_k \rangle$ must be positive so $\mu < \varepsilon_k$ for all $\vec{k}$. Since the smallest $\varepsilon_{\vec{k}}$ is at $\vec{k}=0 \Rightarrow \mu < 0$
* $T \to 0$ $\quad n_{\vec{k}} \to 0$ except $\vec{k}=0$, $\varepsilon_{\vec{k}} = \mu = 0$, all boson occupy the lowest energy level. Due to this fact, at finite T, special care has to

taken when convert $\sum_{\vec{k}}$ to integration.

Specifically, the sum over microstates would be

$$\sum_{\vec{r}}\sum_{\vec{k}} \to V\int \frac{d^3k}{(2\pi)^3} = \int_0^\infty g(\varepsilon)\,d\varepsilon$$

where the density of states $g(\varepsilon) = \frac{2\pi V}{h^3}(2m)^{3/2}\varepsilon^{1/2}$ goes to zero for $\varepsilon = 0$. In other words, the above integral approximation to $\sum_{\vec{k}}$ is very bad. To fix this, we explicitly account for $\vec{k}=0$ -term

$$\sum_{\vec{k}} \to (\vec{k}=0) + \int \frac{d^3k}{(2\pi)^3}$$

The grand potential, thus, is

$$\Omega(T,V,z) = -kT\ln\mathcal{Z} = kT\sum_k \ln\left(1-e^{-\beta\varepsilon}z\right)$$

$$= kT\ln(1-z) + V\int\frac{d^3\vec{k}}{(2\pi)^3}\ln\left(1-ze^{-\beta\varepsilon}\right)\cdot kT$$

$$= kT\ln(1-z) - \frac{2}{3}kT\int_0^\infty d\varepsilon\,\frac{g(\varepsilon)\cdot\varepsilon}{z^{-1}e^{\beta\varepsilon}-1}$$

and

$$N = \sum_{\vec{k}} n_{\vec{k}} = \sum_{\vec{k}}\frac{1}{z^{-1}e^{\beta\varepsilon_{\vec{k}}}-1} = \frac{1}{z^{-1}-1} + \int_0^\infty d\varepsilon\,\frac{g(\varepsilon)}{z^{-1}e^{\beta\varepsilon}-1}$$

$$\equiv N_0 + N_e$$

N_0 : # of bosons in ground state,
N_e : # of bosons is excited state.

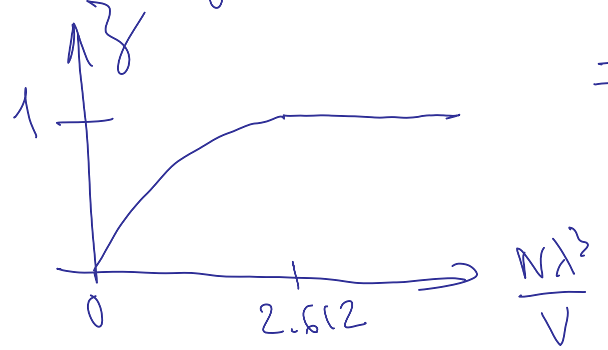
Define a function $g_n(z) = \frac{1}{(n-1)!} \int_0^\infty \frac{x^{n-1}\,dx}{z^{-1}\exp(x) - 1}$

$$\Omega = kT \ln(1-z) - \frac{V}{\lambda^3} g_{5/2}(z)\, kT$$

$$N = \frac{z}{1-z} + \frac{V}{\lambda^3} g_{3/2}(z)$$

if one plots $g_{3/2}(z)$ as function of z, one sees that it has a maximum at $z = 1$ and $N_e^{max} = \frac{V}{\lambda^3} g_{3/2}(1)$
$\simeq (V/\lambda^3)\, 2.612 \sim V \cdot T^{3/2}$

at very small T, $N_e^{max} \ll N$, $z = 1$ and $N \sim N_0$
at very large T, $N_e^{max} \gg 1$, $z < 0$ and $N_0 = 0$



$\Rightarrow$ there's a critical temperature T_c where $N_e^{max} = N$

$$\Rightarrow T_c = \left(\frac{N}{V}\right)^{2/3} \frac{h^2}{2\pi m \left(g_{3/2}(1)\right)^{2/3}}$$

$T < T_c$ | $z = 1$, $N_0 > 0$. There's a <u>macroscopic</u> occupation of single-particle state $\varepsilon_{\vec{k}=0}$. This is Bose-Einstein condensation.

$T > T_c$, $N_0 = 0$ and z is obtained from $N_e(z) = N$
or $\frac{N\lambda^3}{V} = g_{3/2}(z)$

Pressure

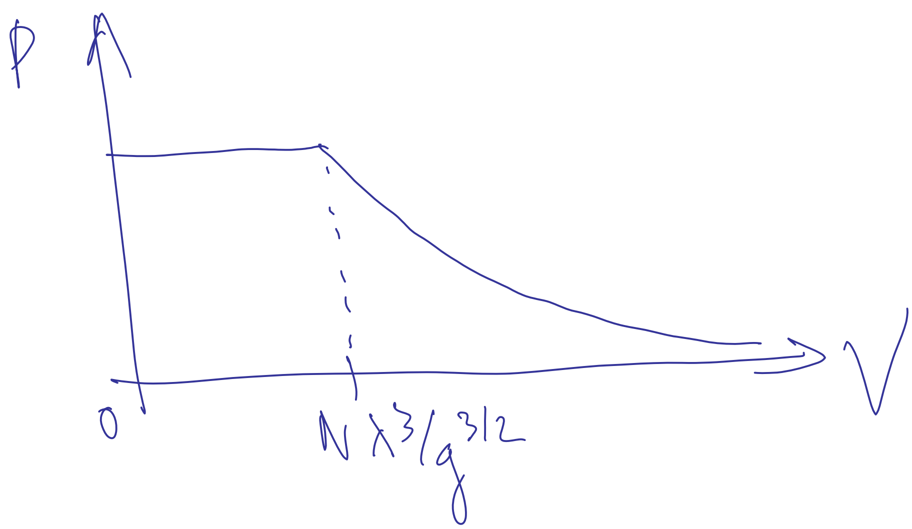
The particles with $\vec{k}=0$ do not contribute

$$P = -\frac{\partial \Omega}{\partial V} = kT \, \frac{g_{5/2}(z)}{\lambda^3}$$

$T<T_c$ $\quad z=1 \qquad P = \frac{g_{5/2}(1)}{\lambda^3}\cdot kT$ independent of V or N

$T\to\infty$, $\quad g_{5/2}(z) \simeq g_{3/2}(z) \simeq z$

So $\quad P = kT\,N/V \quad \Rightarrow$ classical ideal gas



Specific heat:

$$E = -\frac{\partial \ln \mathcal{Z}}{\partial \beta}\Big|_{z,V} = \frac{3}{2} kT \, \frac{V}{\lambda^3} g_{5/2}(z) = \frac{3}{2} PV$$

$$C_V = \frac{\partial E}{\partial T}\Big|_{V,N}$$

$T<T_c$: $\quad C_V = \frac{15}{4}\, g_{5/2}(1) \cdot \frac{Vk}{\lambda^3} \sim T^{3/2}$

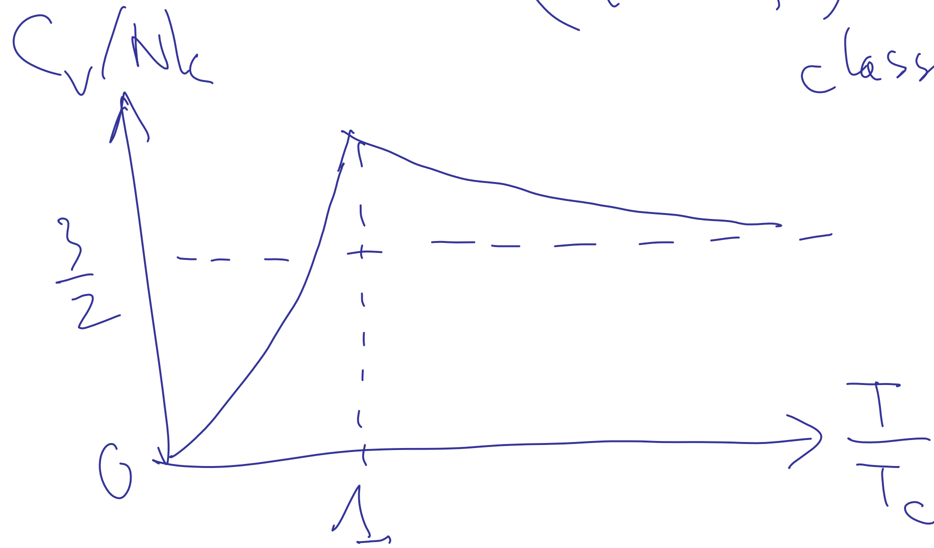
$T > T_c$: $C_V = \left(\frac{15}{4} \frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4} \frac{g_{3/2}(z)}{g_{1/2}(z)} \right) Nk$

(See textbook page 323 for derivation)

for $T \to \infty$, $g_{5/2}(z) \simeq g_{3/2}(z) \simeq z$

$$C_V = \left(\frac{15}{4} - \frac{9}{4} \right) Nk = \frac{3}{2} Nk$$

classical ideal gas



$C_V \sim T^{3/2}$ because at finite T, there's an occupation of states upto k_{max} such that $\frac{\hbar^2 k_{max}^2}{2m} \sim Tk$

the number of states is $\frac{k_{max}^3}{(2\pi)^3}$. So $E \sim V \cdot \frac{k_{max}^3}{(2\pi)^3} \cdot kT$

$\sim T^{5/2}$

$\Rightarrow C_V \sim T^{3/2}$

This is in contrast to Fermion $C_V \sim T$ because only particles near Fermi level are excited $E \sim V \frac{k_F^2}{(2\pi)^2} \cdot \frac{kT}{\varepsilon_F} \cdot kT$