Statistical Mechanics

Assignment #7

Due : 04/11/09 at 5pm

Problem 1, electrons in metal

Consider an ideal quantum gas of Fermi particles at a temperature T.

(a) Write the probability $p(n)$ that there are n particles in a given single particle state as a function of the mean occupation number, $< n >$.

(b) Find the root-mean-square fluctuation $< (n- < n >)^2 >^{1/2}$ in the occupation number of a single particle state as a function of the mean occupation number $< n >$. Sketch the result.

(c) $g(E)$ is the density of states in a metal, E_F is the Fermi energy. Give an expression for the total number of electrons in the system at $T = 0$ in terms of E_F and $g(E_F)$.

(d) Give an expression of the total number of electrons at $T > 0$ in terms of the chemical potential μ and $g(E)$, you don't need to evaluate the integral.

Reminder: Density of state g(ϵ) is defined the same way as in classical Stat Mech. It is the number of microstates with energy in the interval ϵ and $\epsilon + d\epsilon$. In other words, for quantum ideal gas, the sum over all microstates $\sum_{\vec{k}}$ can be replaced by the sum over all energy spectrum. $\int g(\epsilon)d\epsilon$.

Problem 2, electrons in semiconductor

(a) For a system of electrons, assumed non-interacting, show that the probability of finding an electron in a state with energy Δ above the chemical potential μ is the same as the probability of finding an electron absent from a state with energy Δ below μ at any given temperature T.

(b) Suppose that the density of states $g(\epsilon)$ is given by

$$g(\epsilon) = \begin{cases} a(\epsilon - \epsilon_g)^{1/2} & \epsilon > \epsilon_g, \\ 0 & 0 < \epsilon < \epsilon_g, \\ b(-\epsilon)^{1/2} & \epsilon < 0. \end{cases}$$

At $T = 0$ all states with $\epsilon < 0$ are occupied while the other states are empty. Now for $T > 0$, some states with $\epsilon > 0$ will be occupied while some states with $\epsilon < 0$ will be empty. If $a = b$, where is the position of μ? For $a \neq b$, write down the mathematical equation for the determination of μ and discuss qualitatively where μ will be if $a > b$? $a < b$?

(c) If there is an excess of n_d electrons per unit volume than can be accomodated by the states with $\epsilon < 0$, what is the equation for μ for $T = 0$? How will μ shift as T increases?

Problem 3: Debye theory of solid

A solid made of N particles vibrating around their equilibrium positions. The Hamiltonian describing such vibrations can be written as the sum of Hamiltonians of 3N harmonic oscilators with frequencies $\omega = c\vec{k}$ where c is the velocity of sound propagation in the solid. Each quanta of such oscillator is called a phonon. Phonons are bosonic particles.

(a) Calculate the density of states $g(\omega)$ of the oscillators.

(b) Since there're a total of 3N oscilators, there's an upper cut-off frequency ω_c such that $\int_0^{\omega_c} g(\omega)d\omega = 3N$. Calculate this cut-off frequency. This is called Debye frequency.

(c) We can use the ideal gas of phonons to describe the solid instead of real lattice particles (ideal

gas system is always easier to describe than a solid of strong interacting particles). Because phonons are not real, their total number is not a conserved quantity (higher temperature, more phonons; low temperature, less phonons). This means that the chemical potential of phonon is always zero and their fugacity $z = 1$. Write down the average occupation number of the oscillators $< n_{\vec{k}} >$.
(d)Remind that the energy of quantum oscillators is $\epsilon_{\vec{k}} = \hbar\omega(\frac{1}{2} + < n >)$. Obtain an expression for the total energy, specific heat of the phonon gas. Show that at low temperature, $C_V \propto T^3$ while at high temperature C_V is a constant (same as that of classical ideal gas).

Problem 4
Consider a photon gas enclosed in a volume V and in equilibrium at temperature T. The photon is massless particle, so that, $\varepsilon = \hbar kc$. Photons are quanta of electromagnetic radiations. They are very similar to the gas of phonons in problem 3 (zero chemical potential). However, there's no upper cut-off frequency, no zero-point motion of the oscillators. c is the velocity of light.
(a) Determine how the number of photons in the volume depends upon the temperature.
(b) One may write the energy density in the form

$$\frac{\bar{E}}{V} = \int_0^\infty \rho(\varepsilon) d\varepsilon.$$

Determine the form of $\rho(\varepsilon) = g(\varepsilon)\varepsilon/V$, the spectral density of the energy.
(c) What is the temperature dependence of the energy $\bar{E}$?
(d) Calculate the pressure of the gas. How does it depend on the temperature?

Problem 5 In grand-canonical formulation, the energy is related to the grand canonical partition function by

$$E = -\frac{\partial}{\partial \beta} \ln \mathcal{Z}$$

where the derivative is taken at constant fugacity z and volume V. What would the expression be if the derivative is taken at constant chemical potential μ and volume V instead?