Solutions to various problems/examples shown in class

I. *Normalization of Gaussian probability distribution*

The Gaussian probability distribution has the form

$$P(r) = A\exp\left[-(x-x_0)^2/2\sigma^2\right] \quad .$$

The normalization condition reads

$$\int_{-\infty}^{\infty} P(x)\,dx = 1 \quad .$$

To perform this integration, you need to know a very important formula (you can look up this formula from any integration table).

$$\boxed{\int_{-\infty}^{\infty} dy \exp(-y^2) = \sqrt{\pi}} \quad .$$

Using this formula, we get

$$\int_{-\infty}^{\infty} dy \exp(-by^2) = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{b}} \exp(-z^2) = \sqrt{\frac{\pi}{b}} \quad , \qquad (1)$$

where in the 1st identity, we change the integration variable from y to $z = \sqrt{b}\,y$ (correspondingly, dy is replaced by $dz/\sqrt{b}$).

Back to our normalization integration,

$$1 = \int_{-\infty}^{\infty} P(x)\,dx = \int_{-\infty}^{\infty} A\,e^{-(x-x_0)^2/2\sigma^2}\,dx = A\int_{-\infty}^{\infty} dy\,e^{-y^2/2\sigma^2} = A\sqrt{\frac{\pi}{1/2\sigma^2}} = A\sqrt{2\pi}\,\sigma \quad .$$

Therefore, $A = 1/\sqrt{2\pi}\,\sigma$, and the properly normalized Gaussian probability distribution is

$$\boxed{P(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right]} \quad .$$

II. *Example 3.1*

We throw two dices. What is the probability *P* that the sum of the dices equal 6.

To have the sum of dices equal 6, the following event has to occur: (dice 1 has face 1 AND dice 2 has face 5) OR (dice 1 has face 2 AND dice 2 has face 4) OR (dice 1 has face 3 AND dice 2 has face 3) OR (dice 1 has face 4 AND dice 2 has face 2) OR (dice 1 has face 5 AND dice 2 has face 1).

Using addition and multiplication rules for probability, AND corresponds to a multiplication of probabilities, OR corresponds to addition of probabilities, we has

P = P(1,5) + P(2,4) + P(3,3) + P(4,2) + P(5,1)
= P(1) P(5) + P(2) P(4) + P(3) P(3) + P(4) P(2) + P(5) P(1)

Since all the probabilities P(i) = 1/6 for i=1,2,3,4,5,6, we get the final result

P = 5 (1/6) (1/6) = 5/36.

III. *Example 3.2*

Shoot a arrow at a target. Wind currents shift the arrow up-down, left-right randomly. The proability distribution of the point where the arrow hits obeys a Gaussian statistic in x and y direction with spread σ. Calculate the probability P(r)dr that the arrow hits a point at distance r from the bull's eye.

Solution:

The probability P(x)dx that the x-component of the point where the arrow hits obeys Gaussian distribution:

$$P(x)dx=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}} .$$

Similarly

$$P(y)dy=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{y^2}{2\sigma^2}} .$$

Since the x and y directions are independent, the probability that the arrow hits point (x,y) within a distance dx and dy respectively is the multiplication of the probabilities P(x)dx and P(y)dy.

$$P(x,y)\,dx\,dy=P(x)\,dx\times P(y)\,dy=\frac{1}{2\pi\sigma^2}\exp\left[-\frac{x^2+y^2}{2\sigma^2}\right]dx\,dy\equiv\frac{1}{2\pi\sigma^2}\exp\left[-\frac{r^2}{2\sigma^2}\right]d^2r .$$

Since there are many points within a distance r from the bull's-eye, the probability that the arrow hits a point withing distance r is equal to the sum P(x,y) dx dy over all the points. In other words,

$$P(r)\,dr=\frac{1}{2\pi\sigma^2}\exp\left[-\frac{r^2}{2\sigma^2}\right]dS .$$

where dS is the area of the ring between distance r and r+dr from the center.

$$dS=\pi(r+dr)^2-\pi r^2\approx 2\pi r\,dr .$$

Finally,

$$P(r)\,dr=\frac{1}{2\pi\sigma^2}\exp\left[-\frac{r^2}{2\sigma^2}\right]2\pi r\,dr .$$

Example 3.2'

Calculate the accumulative probability $P_{acc}(r)$ that the arrow will land <u>within</u> a distance r from the bull's eye.

Solution: following the addition rule, $P_{acc}(r)$ is the sum of the probabily P(r') dr' that the arrow lands with $0 \le r' \le r$. Using the result of example 3.2, we have

$$P_{acc}(r)=\int_0^r P(r')\,dr'=\int_0^r \frac{1}{2\pi\sigma^2}\exp\left[-\frac{r'^2}{2\sigma^2}\right]2\pi r'\,dr'=\int_0^r \exp\left[-\frac{r'^2}{2\sigma^2}\right]d\left(\frac{r'^2}{2\sigma^2}\right) .$$

Changing the integration variable to $y = r'^2/2\sigma^2$, we obtain the final result:

$$P^{acc}(r)=\int_0^{\frac{r^2}{2\sigma^2}} e^{-y}\,dy=1-e^{-r^2/2\sigma^2} .$$

Note: The accumulative probability function is a monotonously increasing function from 0 at $r = 0$ to 1 at $r = \infty$. The later is obviously the consequence of the fact that the probability that the arrow hits ANY points on the target is 100% (the normalization condition).

III. Example 3.3

Generalize example 3.2 to calculate the probability distribution P(r) dr for a 3-dimensional random vector **u**. Assuming **u** obeys Gaussian statistic in x, y and z directions.

Solution:

This is a homework problem. See homework solution when posted.
The answer is

$$P(r)dr=\left(\frac{1}{\sqrt{2\pi\sigma}}\right)^3 e^{-r^2/(2\sigma^2)}\times 4\pi r^2 dr$$

IV. Example 3.4

Calculate the variance of the Gaussian probability distribution.

Solution:

By definition, variance(x) = <(x –<x>)^2>. Since <x> = x_0 (verify this!), we have,

$$variance(x)=\int_{-\infty}^{\infty}(x-\langle x\rangle)^2 P(x)dx=\int_{-\infty}^{\infty}(x-x_0)^2\frac{1}{\sqrt{2\pi\sigma}}\exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right]dx=\frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty}y^2e^{-y^2}dy \quad .$$

where in the last identity, we change the variable $y=\frac{x-x_0}{\sqrt{2}\sigma}$ and, correspondingly, $dx=dy\sqrt{2}\sigma$.

To calculate the integration above, we use a differentiation trick. Starting from Equation (1),

$$\int_{-\infty}^{\infty}dy\,e^{-by^2}=\sqrt{\frac{\pi}{b}} \quad .$$

We differentiate the two sides of this identity with respect to *b*. This gives,

$$\int_{-\infty}^{\infty}dy(-y^2)e^{-by^2}=-\frac{1}{2}\sqrt{\frac{\pi}{b^3}} \quad .$$

Setting b=1 in the above formula and substituting them into the expression for variance(x), we get

$$variance(x)=\sigma^2 \quad .$$

V. Example 3.5

What is the average speed of a gas molecule in the room.

Solution: Using the formula relating average kinetic energy to temperature, $\frac{3}{2}k_BT=\frac{1}{2}m\langle v^2\rangle$.

Since nitrogen is the dominant molecules of air, we use for *m* the atomic mass of N_2 molecule (28Da). Put in the numbers, (T~300ºK), we obtain v ~ 500 m/s or 1120 miles/hour. It's supersonic.

VI. Example 3.6

Calculate the average and most probable speed of ideal gas

Solution: From the result of Example 3.3,

$$P(v)dv = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^3 e^{-v^2/(2\sigma^2)} \times 4\pi v^2 dv \quad .$$

The most probable speed, u_m, corresponds to the maximum of P(v). This maximum is obtained by setting the first derivative equal to zero

$$0 = \left.\frac{\partial P(v)}{\partial v}\right|_{v=v_m} = \frac{4\pi}{(\sqrt{2\pi}\sigma)^3} \left[\frac{-v_m^3}{\sigma} + 2v_m\right] e^{-v^2/(2\sigma^2)} \quad .$$

Thus

$$v_m = \sigma/\sqrt{2} \quad .$$

The average speed is calculated from

$$\langle v \rangle = \int_0^\infty v P(v) dv = \frac{4\pi}{(\sqrt{2\pi}\sigma)^3} \int_0^\infty dv\, v \times v^2 e^{-v^2/2\sigma^2} \quad .$$

Making the substitution, $\bar{v} = v^2/2\sigma^2$, we get

$$\langle v \rangle = \frac{4\sigma}{\sqrt{2\pi}} \int_0^\infty d\bar{v}\, \bar{v} e^{-\bar{v}} = \frac{4\sigma}{\sqrt{2\pi}} \quad .$$

Obviously the average speed, <v> is greater than the v_m. This is because the curve P(r) is not symmetric, with more area under the curve on the right side of the maximum.