Midterm 2, 11/05/07

Problem 1: (10 points)

Consider the following liquids: pentane, glycerol, crude oil, and liquid helium (but not superfluid helium). Their viscosities are 0.24×10^{-3}, 1.5, 3×10^{-2}, 25×10^{-9} Pa·s respectively.

(a) How fast would a bacterium of diameter 1 μm have to swim through each of these liquids, in order to attain a Reynolds number of 1?

(b) Consider again the bacterium of diameter 1 μm, and traveling at a speed of 10 μm/s through each of the above liquids. If it suddenly stops swimming, how far will it travel (in each liquid) before it stops? (You may assume the bacterium is mostly made of water, regardless of what medium it is swimming in.)

Problem 2: (15 points)

A zipper has N links. Each link has a state in which it is closed with energy 0 and a state in which it is open with energy ε. We require that the zipper can be unzip from one side (say, from the left). Each microstate is characterized by n left open links (and $N\!-\!n$ right closed links), n=0,...N. This model is sometime used to study DNA unzipping.

(a) Find the partition function

(b) How the free energy behaves at very hight temperature? at very low temperature ?

(c) Write down an expression for the average number of open links in terms of ε and N and T. (hint $\langle n \rangle = \sum_{n=0}^{N} P(n) n$). How this average number behaves at very hight temperature? at very low temperature ?

Problem 3: (5 points)

Viral DNA molecule is tightly bend inside a bacteriophage (a virus) creating a huge pressure inside the viral capsid. When the phage attaches to a host cell, this stored pressure ejects the DNA into the host. Experiments show that in the presence of a finite concentration of PEG8000 molecule (8000 gram/mole molecular weight), the amount of ejected DNA is reduced. This is because PEG8000 molecules are large and cannot enter the viral capsid, hence they exert an osmotic pressure on the capsid countering the pressure stored inside the capsid.

Accurate measurement indicates that in a water solution of PEG8000 at 30% weight ratio (weight of PEG vs total weight), DNA ejection is completely inhibited at temperature T=300K. Find the pressure stored inside viral capsid (You can assume the density of solution remains 1kg/litter).

Solution

Problem 1

(a) $\Re = R v \rho / \eta$. Just for this particular question, ρ is not given so we set $\rho = 10^3\, kg/m^3$, same as water. Substitute $R = 0.5\,\mu m = 0.5\times10^{-6} m$ and $\Re = 1$, we get $v = 0.48,\ 3\times10^3,\ 60, \text{and}\ 5\times10^{-5}$ m/s respectively.

(b) $m\dot{v} = -f_{vis} = -6\pi\eta R v$, so $v(t) = v(t=0)\exp(-t/\tau)$ where $\tau = m/6\pi\eta R$. The distance the bacterial travel after it stops swimming is $L = \int_0^\infty v(t)\,dt = v_0\tau = v_0(4\pi R^3/3)\rho/6\pi\eta R = 2v_0\rho R^2/9\eta$. Put in the numbers we get $L = 2.3\times10^{-12},\ 3.7\times10^{-16},\ 1.8\times10^{-14},\ 2.2\times10^{-8}\,\text{m}$ respectively.

Problem 2:

(a) $Z = \sum \exp(-E_i/k_B T)$. The sum is over all microstates. In our case, each microstate is characterized by n, energy $E(n) = n\epsilon$. So

$$Z = \sum_{n=0}^{N} \exp(-n\epsilon/k_B T) = \frac{1-x^{N+1}}{1-x} \quad \text{where} \quad x = \exp(-\epsilon/k_B T) \ .$$

(b) $F = -k_B T \ln Z$. For very high T, x → 1, $F \to -k_B T \ln(N+1)$.
For very low T, x → 0, $F \to -k_B T x$.

(c) $$\langle n\rangle = \sum_0^N P(n) n = \sum_0^N n \exp(-n\epsilon/k_B T) Z^{-1} = -Z^{-1}\frac{d}{d(\epsilon/k_B T)}\sum_0^N \exp(-n\epsilon/k_B T)$$

$$\langle n\rangle = -\frac{d}{d(\epsilon/k_B T)}\ln Z = \frac{x}{1-x} - \frac{(N+1)x^{N+1}}{1-x^{N+1}} \ .$$

For very high T, x → 1, <n> → N (all links are open).
For very low T, x → 0, <n> → 0 (all links are close).

Problem 3:

The osmotic pressure is $P = c k_B T$.

In one litter of water, there's 300g of PEG8000, or 300/8000 = 0.0375 mole = 0.0375 x 6 x 10^{23} molecules of PEG. The PEG concentration is thus

$$c = 0.0375\times6\times10^{23}/10^{-3}\,\text{m}^{-3} = 225\times10^{23}\,\text{m}^{-3} \ .$$

The osmotic pressure is then, $P = 225\times10^{23}\times4.1\times10^{-21} = 92000\,\text{Pa}.$