

PROBLEMS

5.1 *Friction versus dissipation*

Gilbert says: You say that friction and dissipation are two manifestations of the same thing. So high viscosity must be a very dissipative situation. Then why do I get beautifully ordered, laminar motion only in the *high-viscosity* case? Why does my ink blob miraculously reassemble itself only in this case?

Sullivan: Um, uh ...

Help Sullivan out.

5.2 *Density profile*

Finish the derivation of particle density in an equilibrium colloidal suspension (begun in Section 5.1.1) by finding the constant prefactor in Equation 5.1. That is, find a formula for the equilibrium number density $c(x)$ of particles with net weight $m_{\text{net}}g$ as a function of the height x . The total number of particles is N ; the height of the test tube is h and its cross sectional area is A .

5.3 *Archibald method*

Sedimentation is a key analytical tool in the lab for the study of big molecules. Consider a particle of mass m and volume V in a fluid of mass density ρ_m and viscosity η .

- a. Suppose a test tube is spun in the plane of a wheel, pointing along one of the "spokes." The artificial gravity field in the centrifuge is not uniform; rather, it is stronger at one end of the tube than the other. Hence the sedimentation rate will not be uniform either. Suppose that one end lies a distance r_1 from the center, and the other end is at $r_2 = r_1 + \ell$. The centrifuge is spun at angular frequency ω . Adapt the formula $v_{\text{drift}} = gs$ (Equation 5.3 on page 160) to find an analogous formula for the drift speed in terms of s in the centrifuge case.

Eventually, sedimentation will stop and an equilibrium profile will emerge. It may take quite a long time for the whole test tube to reach its equilibrium distribution. In that case, Equation 5.2 on page 160 is not the most convenient way to measure the mass parameter m_{net} . The **Archibald method** uses the fact that the *ends* of the test tube equilibrate rapidly, as follows.

- b. There can be no flux of material through the ends of the tube. Thus, the Fick-law flux must cancel the flux you found in (a). Write down two equations expressing this statement at the two ends of the tube.
- c. Derive the following expression for the mass parameter in terms of the concentration and its gradient at one end of the tube:

$$m_{\text{net}} = (\text{stuff}) \times \left. \frac{dc}{dr} \right|_{r=r_1},$$

and a similar formula for the other end, where (stuff) is some factors that you are to find. The concentration and its gradient can be measured photometrically in

the lab, thus allowing a measurement of m_{net} long before the whole test tube has come to equilibrium.

5.4 Coasting at low Reynolds

The chapter asserted that tiny objects stop moving at once when we stop pushing them. Let's see.

- Consider a bacterium, idealized as a sphere of radius $1 \mu\text{m}$, propelling itself at $1 \mu\text{m s}^{-1}$. At time zero, the bacterium suddenly stops swimming and coasts to a stop, following Newton's Law of motion with the Stokes drag force. How far does it travel before it stops? Comment.
- Our discussion of Brownian motion assumed that each random step was independent of the previous one; thus, for example, we neglected the possibility of a residual drift speed left over from the previous step. In the light of (a), would you say that this assumption is justified for a bacterium?

5.5 Blood flow

Your heart pumps blood into your aorta. The maximum flow rate into the aorta is about $500 \text{ cm}^3 \text{ s}^{-1}$. Assume that the aorta has diameter 2.5 cm, that the flow is laminar (not very accurate), and that blood is a Newtonian fluid with viscosity roughly equal to that of water.

- Find the pressure drop per unit length along the aorta. Express your answer in SI units. Compare the pressure drop along a 10 cm section of aorta with atmospheric pressure (10^5 Pa).
- How much power does the heart expend just pushing blood along a 10 cm section of aorta? Compare your answer with your basal metabolic rate, about 100 W, and comment.
- The fluid velocity in laminar pipe flow is zero at the walls of the pipe and maximum at the center. Sketch the velocity as a function of distance r from the center. Find the velocity at the center. [Hint: The total volume flow rate, which you are given, equals $\int v(r)2\pi r dr$.]

5.6 T_2 Kinematic viscosity

- Although the kinematic viscosity ν has the same dimensions L^2/T as any other diffusion constant, its physical meaning is quite different from that of D , and its numerical value for water is quite different from the value of D for self-diffusion of water molecules. Find the value of ν from η and compare with D .
- Still, these values are related. Show, by combining Einstein's relation and the Stokes formula, that taking the radius R of a water molecule to be about 0.2 nm leads to a satisfactory order-of-magnitude prediction of ν from D , R , and the mass density of water.

5.7 T_2 No going back

Section 5.2.3 argued that the motion of a gently sheared, flat layer would retrace its history if we reverse the applied force. When the force is large, so that we cannot ignore the inertial term in Newton's Law of motion, where exactly does the argument fail?

Returning to myoglobin, it may seem as though sedimentation is not a very useful tool for protein analysis. But the scale height depends not only on properties of the protein and solvent but also on the acceleration of gravity, g . Artificially increasing g with a centrifuge can reduce z_* to a manageably small value; indeed, laboratory centrifuges can attain values of g' up to around 10^6 m s^{-2} , making protein separation feasible.

To make these remarks precise, first note that, when a particle gets whirled about at angular frequency ω , a first-year physics formula gives its centripetal acceleration as $r\omega^2$, where r is the distance from the center.

**Your
Turn
5B**

Suppose you didn't remember this formula. Show how to guess it by dimensional analysis, knowing that angular frequency is measured in radians/s.

Suppose that the sample is in a tube lying in the plane of rotation, so that its long axis points radially. The centripetal acceleration points inward, toward the axis of rotation, so there must be an inward-pointing force, $f = -m_{\text{net}}r\omega^2$, causing it. This force can only come from the frictional drag of the surrounding fluid as the particle drifts slowly outward. Thus, the drift velocity is given by $m_{\text{net}}r\omega^2/\zeta$ (see Equation 4.12 on page 119). Repeating the argument that led to the Nernst relation (Section 4.6.3 on page 139) now gives the drift flux as $cv_{\text{drift}} = cm_{\text{net}}r\omega^2 D/k_{\text{B}}T$, where $c(r)$ is the number density. In equilibrium, this drift flux is canceled by a diffusive flux, given by Fick's law. We thus find that, in equilibrium,

$$j = 0 = D\left(-\frac{dc}{dr} + \frac{r\omega^2 m_{\text{net}}}{k_{\text{B}}T}c\right),$$

a result analogous to the Nernst–Planck formula (Equation 4.24 on page 140). To solve this differential equation, divide by $c(r)$ and integrate:

$$c = \text{const} \times e^{m_{\text{net}}\omega^2 r^2/(2k_{\text{B}}T)}. \quad (\text{sedimentation equilibrium, centrifuge}) \quad (5.2)$$

5.1.2 The rate of sedimentation depends on solvent viscosity

The drift velocity $v_{\text{drift}} = m_{\text{net}}g/\zeta$ isn't an intrinsic property of the particle, because it depends on the strength of gravity, g . To get a quantity that we can tabulate for various particle types (in given solvents), we instead define the **sedimentation coefficient**

$$s = v_{\text{drift}}/g \equiv m_{\text{net}}/\zeta. \quad (5.3)$$

Measuring s and looking in a table thus gives a rough-and-ready particle identification. The quantity s can be interpreted as the time required for a particle to come to terminal velocity. It is sometimes expressed in units of **svedbergs**; a svedberg by definition equals 10^{-13} s .

What determines the sedimentation coefficient s ? Surely sedimentation will be slower in a “thick” liquid like honey than in a “thin” one like water. That is, we expect the viscous friction coefficient ζ for a single particle in a fluid to depend not only on the size of the particle but also on some intrinsic property of the fluid, called the viscosity. In fact, Section 4.1.4 already quoted an expression for ζ , namely, the Stokes formula, $\zeta = 6\pi\eta R$, for an isolated, spherical particle of radius R .

**Your
Turn
5C**

- Work out the dimensions of η from the Stokes formula. Show that they can be regarded as those of pressure times time and that, hence, the SI units for viscosity are Pa s.
- Your Turn 5A raised a paradox: The equilibrium formula you found suggested that milk should separate, and yet we don’t normally observe this happening. Use the Stokes formula to estimate how long it takes for fat globules in homogenized milk to drift a distance equal to the size of a milk bottle. Then compare homogenized milk with raw milk (which has fat droplets up to about $5\ \mu\text{m}$ in diameter), and comment.

It’s worth memorizing the value of η for water at room temperature:¹ $\eta_w \approx 10^{-3}\ \text{kg m}^{-1}\ \text{s}^{-1} = 10^{-3}\ \text{Pa s}$.

We can use the preceding remarks to look once again at the sizes of polymer coils. Let’s suppose that a particular type of polymer forms random coils, with radius given by a constant times some power of the molecular mass: $R \propto m^p$. We’d like to verify this claim, and extract the value of the scaling exponent p , from an experiment. Then we’ll compare the result to the prediction from random-walk theory, which is that $p = \frac{1}{2}$ (Idea 4.17 on page 123).

Combining Equation 5.3 with the Stokes formula gives $s = (m - V\rho_m)/(6\pi\eta R)$. Assuming that the polymer displaces a volume of water proportional to the number of monomers yields $s \propto m^{1-p}$. Figure 4.7b on page 123 shows that our prediction $p = \frac{1}{2}$ indeed is roughly true. (More precisely, for one particular polymer/solvent combination Figure 4.7a gives the scaling exponent for R as $p = 0.57$. Figure 4.7b gives the exponent for s as 0.44, which is quite close to $1 - p$.)

5.1.3 It’s hard to mix a viscous liquid

Section 5.2 will argue that, in the nanoworld of cells, ordinary water behaves as a very viscous liquid. Because most people have made only limited observations in this world, it’s worthwhile to pause first and notice some of the spooky phenomena that happen there.

Pour a few centimeters of clear corn syrup into a clear cylindrical beaker or wide cup. Set aside some of the syrup and mix it with a small amount of ink to serve as a marker. Put a stirring rod in the beaker, then inject a small blob of marked syrup

¹Some authors express this result in units of **poise**, defined as $\text{erg s/cm}^2 = 0.1\ \text{Pa s}$; thus η_w is about one centipoise. Values of η for other biologically relevant fluids appear in Table 5.1 on page 165.