Problem 1.32

$\nabla\cdot\mathbf{v} = y + 2z + 3x$

$$\begin{aligned}
\int(\nabla\cdot\mathbf{v})d\tau &= \int(y+2z+3x)\,dx\,dy\,dz = \iint\left\{\int_0^2(y+2z+3x)\,dx\right\}dy\,dz \\
&\qquad\qquad\qquad\hookrightarrow \left[(y+2z)x + \tfrac{3}{2}x^2\right]_0^2 = 2(y+2z)+6 \\
&= \int\left\{\int_0^2(2y+4z+6)dy\right\}dz \\
&\qquad\hookrightarrow \left[y^2+(4z+6)y\right]_0^2 = 4+2(4z+6) = 8z+16 \\
&= \int_0^2(8z+16)dz = (4z^2+16z)\Big|_0^2 = 16+32 = \boxed{48.}
\end{aligned}$$

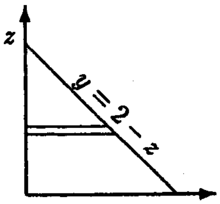
Numbering the surfaces as in Fig. 1.29:

(i) $d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}, x = 2.$ $\mathbf{v}\cdot d\mathbf{a} = 2y\,dy\,dz.$ $\int\mathbf{v}\cdot d\mathbf{a} = \iint 2y\,dy\,dz = 2y^2\Big|_0^2 = 8.$
(ii) $d\mathbf{a} = -dy\,dz\,\hat{\mathbf{x}}, x = 0.$ $\mathbf{v}\cdot d\mathbf{a} = 0.$ $\int\mathbf{v}\cdot d\mathbf{a} = 0.$
(iii) $d\mathbf{a} = dx\,dz\,\hat{\mathbf{y}}, y = 2.$ $\mathbf{v}\cdot d\mathbf{a} = 4z\,dx\,dz.$ $\int\mathbf{v}\cdot d\mathbf{a} = \iint 4z\,dx\,dz = 16.$
(iv) $d\mathbf{a} = -dx\,dz\,\hat{\mathbf{y}}, y = 0.$ $\mathbf{v}\cdot d\mathbf{a} = 0.$ $\int\mathbf{v}\cdot d\mathbf{a} = 0.$
(v) $d\mathbf{a} = dx\,dy\,\hat{\mathbf{z}}, z = 2.$ $\mathbf{v}\cdot d\mathbf{a} = 6x\,dx\,dy.$ $\int\mathbf{v}\cdot d\mathbf{a} = 24.$
(vi) $d\mathbf{a} = -dx\,dy\,\hat{\mathbf{z}}, z = 0.$ $\mathbf{v}\cdot d\mathbf{a} = 0.$ $\int\mathbf{v}\cdot d\mathbf{a} = 0.$
$\Rightarrow \int\mathbf{v}\cdot d\mathbf{a} = 8 + 16 + 24 = 48$ ✓

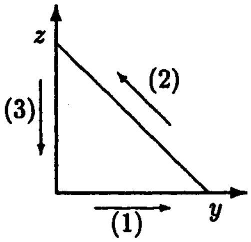
Problem 1.33

$\nabla\times\mathbf{v} = \hat{\mathbf{x}}(0-2y) + \hat{\mathbf{y}}(0-3z) + \hat{\mathbf{z}}(0-x) = -2y\,\hat{\mathbf{x}} - 3z\,\hat{\mathbf{y}} - x\,\hat{\mathbf{z}}.$
$d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}$, if we agree that the path integral shall run counterclockwise. So
$(\nabla\times\mathbf{v})\cdot d\mathbf{a} = -2y\,dy\,dz.$

$$\begin{aligned}
\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} &= \int\left\{\int_0^{2-z}(-2y)dy\right\}dz \\
&\qquad\hookrightarrow y^2\Big|_0^{2-z} = -(2-z)^2 \\
&= -\int_0^2(4-4z+z^2)dz = -\left(4z - 2z^2 + \tfrac{z^3}{3}\right)\Big|_0^2 \\
&= -\left(8-8+\tfrac{8}{3}\right) = \boxed{-\tfrac{8}{3}}
\end{aligned}$$



Meanwhile, $\mathbf{v}\cdot d\mathbf{l} = (xy)dx + (2yz)dy + (3zx)dz$. There are three segments.



(1) $x = z = 0;$ $dx = dz = 0.$ $y: 0 \to 2.$ $\int\mathbf{v}\cdot d\mathbf{l} = 0.$
(2) $x = 0;$ $z = 2 - y;$ $dx = 0,$ $dz = -dy,$ $y: 2 \to 0.$ $\mathbf{v}\cdot d\mathbf{l} = 2yz\,dy.$
$\int\mathbf{v}\cdot d\mathbf{l} = \int_2^0 2y(2-y)dy = -\int_0^2(4y-2y^2)dy = -\left(2y^2 - \tfrac{2}{3}y^3\right)\Big|_0^2 = -\left(8 - \tfrac{2}{3}\cdot 8\right) = -\tfrac{8}{3}.$
(3) $x = y = 0;$ $dx = dy = 0;$ $z: 2 \to 0.$ $\mathbf{v}\cdot d\mathbf{l} = 0.$ $\int\mathbf{v}\cdot d\mathbf{l} = 0.$ So $\oint\mathbf{v}\cdot d\mathbf{l} = -\tfrac{8}{3}.$ ✓

Problem 1.36 $\boxed{r = \sqrt{x^2+y^2+z^2};\quad \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right);\quad \phi = \tan^{-1}\left(\frac{y}{x}\right).}$

Problem 1.37

There are many ways to do this one—probably the most illuminating way is to work it out by trigonometry from Fig. 1.36. The most systematic approach is to study the expression:

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}} = r\sin\theta\cos\phi\,\hat{\mathbf{x}} + r\sin\theta\sin\phi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}}.$$

If I only vary r slightly, then $d\mathbf{r} = \frac{\partial}{\partial r}(\mathbf{r})dr$ is a short vector pointing in the direction of increase in r. To make it a unit vector, I must divide by its length. Thus:

$$\hat{\mathbf{r}} = \frac{\frac{\partial\mathbf{r}}{\partial r}}{\left|\frac{\partial\mathbf{r}}{\partial r}\right|};\ \hat{\boldsymbol{\theta}} = \frac{\frac{\partial\mathbf{r}}{\partial\theta}}{\left|\frac{\partial\mathbf{r}}{\partial\theta}\right|};\ \hat{\boldsymbol{\phi}} = \frac{\frac{\partial\mathbf{r}}{\partial\phi}}{\left|\frac{\partial\mathbf{r}}{\partial\phi}\right|}.$$

$\frac{\partial\mathbf{r}}{\partial r} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}};\ \left|\frac{\partial\mathbf{r}}{\partial r}\right|^2 = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1.$

$\frac{\partial\mathbf{r}}{\partial\theta} = r\cos\theta\cos\phi\,\hat{\mathbf{x}} + r\cos\theta\sin\phi\,\hat{\mathbf{y}} - r\sin\theta\,\hat{\mathbf{z}};\ \left|\frac{\partial\mathbf{r}}{\partial\theta}\right|^2 = r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta = r^2.$

$\frac{\partial\mathbf{r}}{\partial\phi} = -r\sin\theta\sin\phi\,\hat{\mathbf{x}} + r\sin\theta\cos\phi\,\hat{\mathbf{y}};\ \left|\frac{\partial\mathbf{r}}{\partial\phi}\right|^2 = r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi = r^2\sin^2\theta.$

$$\Rightarrow \boxed{\begin{array}{l}\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}.\\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}.\\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}.\end{array}}$$

Check: $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta = \sin^2\theta + \cos^2\theta = 1,$ ✓

$\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = -\cos\theta\sin\phi\cos\phi + \cos\theta\sin\phi\cos\phi = 0,$ ✓ etc.

$\sin\theta\,\hat{\mathbf{r}} = \sin^2\theta\cos\phi\,\hat{\mathbf{x}} + \sin^2\theta\sin\phi\,\hat{\mathbf{y}} + \sin\theta\cos\theta\,\hat{\mathbf{z}}.$

$\cos\theta\,\hat{\boldsymbol{\theta}} = \cos^2\theta\cos\phi\,\hat{\mathbf{x}} + \cos^2\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\cos\theta\,\hat{\mathbf{z}}.$

Add these:

(1) $\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}} = +\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}};$

(2) $\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}.$

Multiply (1) by $\cos\phi$, (2) by $\sin\phi$, and subtract:

$$\boxed{\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}.}$$

Multiply (1) by $\sin\phi$, (2) by $\cos\phi$, and add:

$$\boxed{\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}.}$$

$\cos\theta\,\hat{\mathbf{r}} = \sin\theta\cos\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\cos\theta\sin\phi\,\hat{\mathbf{y}} + \cos^2\theta\,\hat{\mathbf{z}}.$

$\sin\theta\,\hat{\boldsymbol{\theta}} = \sin\theta\cos\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin^2\theta\,\hat{\mathbf{z}}.$

Subtract these:

$$\boxed{\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}.}$$

Problem 1.39

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \tfrac{1}{r^2}\tfrac{\partial}{\partial r}(r^2\,r\cos\theta)+\tfrac{1}{r\sin\theta}\tfrac{\partial}{\partial\theta}(\sin\theta\, r\sin\theta)+\tfrac{1}{r\sin\theta}\tfrac{\partial}{\partial\phi}(r\sin\theta\cos\phi)\\ &= \tfrac{1}{r^2}\,3r^2\cos\theta+\tfrac{1}{r\sin\theta}\,r\,2\sin\theta\cos\theta+\tfrac{1}{r\sin\theta}\,r\sin\theta(-\sin\phi)\\ &= 3\cos\theta+2\cos\theta-\sin\phi = 5\cos\theta-\sin\phi\end{aligned}$$

$$\int(\nabla\cdot\mathbf{v})d\tau = \int(5\cos\theta-\sin\phi)\,r^2\sin\theta\,dr\,d\theta\,d\phi = \int_0^R r^2\,dr\int_0^{\frac{\pi}{2}}\Big[\underbrace{\int_0^{2\pi}(5\cos\theta-\sin\phi)\,d\phi}_{\hookrightarrow 2\pi(5\cos\theta)}\Big]\,d\theta\sin\theta$$

$$= \left(\tfrac{R^3}{3}\right)(10\pi)\underbrace{\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,,d\theta}_{\hookrightarrow \frac{\sin^2\theta}{2}\Big|_0^{\frac{\pi}{2}}=\frac{1}{2}}$$

$$= \boxed{\tfrac{5\pi}{3}R^3.}$$

Two surfaces—one the hemisphere: $d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$; $r = R$; $\phi: 0\to 2\pi$, $\theta: 0\to\frac{\pi}{2}$.

$\int\mathbf{v}\cdot d\mathbf{a} = \int(r\cos\theta)R^2\sin\theta\,d\theta\,d\phi = R^3\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,d\theta\int_0^{2\pi}d\phi = R^3\left(\frac{1}{2}\right)(2\pi) = \pi R^3.$

other the flat bottom: $d\mathbf{a} = (dr)(r\sin\theta\,d\phi)(+\hat{\boldsymbol{\theta}}) = r\,dr\,d\phi\,\hat{\boldsymbol{\theta}}$ (here $\theta = \frac{\pi}{2}$). $r: 0\to R$, $\phi: 0\to 2\pi$.

$\int\mathbf{v}\cdot d\mathbf{a} = \int(r\sin\theta)(r\,dr\,d\phi) = \int_0^R r^2\,dr\int_0^{2\pi}d\phi = 2\pi\frac{R^3}{3}.$

Total: $\int\mathbf{v}\cdot d\mathbf{a} = \pi R^3+\frac{2}{3}\pi R^3 = \frac{5}{3}\pi R^3.$ ✓

Problem 1.44

(a) $\int_{-2}^{2}(2x+3)\frac{1}{3}\delta(x)\,dx = \frac{1}{3}(0+3) = \boxed{1.}$

(b) By Eq. 1.94, $\delta(1-x) = \delta(x-1)$, so $1+3+2 = \boxed{6.}$

(c) $\int_{-1}^{1}9x^2\frac{1}{3}\,\delta(x+\frac{1}{3})\,dx = 9\left(-\frac{1}{3}\right)^2\frac{1}{3} = \boxed{\frac{1}{3}.}$

(d) $\boxed{1 \text{ (if } a>b),\ 0 \text{ (if } a<b).}$

Problem 1.46

(a) $\boxed{\rho(\mathbf{r}) = q\delta^3(\mathbf{r}-\mathbf{r}').}$ Check: $\int\rho(\mathbf{r})d\tau = q\int\delta^3(\mathbf{r}-\mathbf{r}')\,d\tau = q.$ ✓

(b) $\boxed{\rho(\mathbf{r}) = q\delta^3(\mathbf{r}-\mathbf{r}')-q\delta^3(\mathbf{r}).}$

(c) Evidently $\rho(r) = A\delta(r-R)$. To determine the constant A, we require

$Q = \int\rho\,d\tau = \int A\delta(r-R)4\pi r^2\,dr = A\,4\pi R^2.$ So $A = \frac{Q}{4\pi R^2}.$ $\boxed{\rho(r) = \frac{Q}{4\pi R^2}\delta(r-R).}$

Problem 1.48

First method: use Eq. 1.99 to write $J = \int e^{-r}\left(4\pi\delta^3(\mathbf{r})\right)\,d\tau = 4\pi e^{-0} = \boxed{4\pi.}$

Second method: integrating by parts (use Eq. 1.59).

$$\begin{aligned} J &= -\int_{\mathcal{V}}\frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla(e^{-r})\,d\tau+\oint_{\mathcal{S}}e^{-r}\frac{\hat{\mathbf{r}}}{r^2}\cdot d\mathbf{a}.\quad\text{But}\quad\nabla\left(e^{-r}\right) = \left(\frac{\partial}{\partial r}e^{-r}\right)\hat{\mathbf{r}} = -e^{-r}\hat{\mathbf{r}}.\\ &= \int\frac{1}{r^2}e^{-r}4\pi r^2\,dr+\int e^{-r}\frac{\hat{\mathbf{r}}}{r^2}\cdot r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}} = 4\pi\int_0^\infty e^{-r}\,dr+e^{-R}\int\sin\theta\,d\theta\,d\phi\\ &= 4\pi\left(-e^{-r}\right)\Big|_0^\infty+4\pi e^{-R} = 4\pi\left(-e^{-\infty}+e^{-0}\right) = 4\pi.\checkmark\quad(\text{Here } R=\infty,\ \text{so } e^{-R}=0.)\end{aligned}$$

Problem 1.49 (a) $\boldsymbol{\nabla}\cdot\mathbf{F}_1 = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}\left(x^2\right) = \boxed{0}$; $\quad \boldsymbol{\nabla}\cdot\mathbf{F}_2 = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = \boxed{3}$

$$\boldsymbol{\nabla}\times\mathbf{F}_1 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x^2 \end{vmatrix} = -\hat{\mathbf{y}}\,\frac{\partial}{\partial x}\left(x^2\right) = \boxed{-2x\hat{\mathbf{y}}}; \quad \boldsymbol{\nabla}\times\mathbf{F}_2 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \boxed{0}$$

$\boxed{\mathbf{F}_2 \text{ is a gradient; } \mathbf{F}_1 \text{ is a curl}}$ $\quad \boxed{U_2 = \frac{1}{2}\left(x^2 + y^2 + z^2\right)}$ would do ($\mathbf{F}_2 = \boldsymbol{\nabla} U_2$).

For $\mathbf{A}_1$, we want $\left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right) = \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) = 0$; $\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = x^2$. $A_y = \frac{x^3}{3}$, $A_x = A_z = 0$ would do it.

$\boxed{\mathbf{A}_1 = \frac{1}{3}x^2\,\hat{\mathbf{y}}}$ $\quad (\mathbf{F}_1 = \boldsymbol{\nabla}\times\mathbf{A}_1)$. (But these are <u>not</u> unique.)

(b) $\boldsymbol{\nabla}\cdot\mathbf{F}_3 = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$; $\quad \boldsymbol{\nabla}\times\mathbf{F}_3 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \hat{\mathbf{x}}\,(x - x) + \hat{\mathbf{y}}\,(y - y) + \hat{\mathbf{z}}\,(z - z) = 0$

So $\mathbf{F}_3$ can be written as the gradient of a scalar ($\mathbf{F}_3 = \boldsymbol{\nabla} U_3$) and as the curl of a vector ($\mathbf{F}_3 = \boldsymbol{\nabla}\times\mathbf{A}_3$). In fact, $\boxed{U_3 = xyz}$ does the job. For the vector potential, we have

$$\left\{\begin{array}{lll} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = yz, & \text{which suggests} & A_z = \frac{1}{4}y^2z + f(x,z);\; A_y = -\frac{1}{4}yz^2 + g(x,y) \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = xz, & \text{suggesting} & A_x = \frac{1}{4}z^2x + h(x,y);\; A_z = -\frac{1}{4}zx^2 + j(y,z) \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = xy, & \text{so} & A_y = \frac{1}{4}x^2y + k(y,z);\; A_x = -\frac{1}{4}xy^2 + l(x,y) \end{array}\right\}$$

Putting this all together: $\boxed{\mathbf{A}_3 = \frac{1}{4}\left\{x\left(z^2 - y^2\right)\hat{\mathbf{x}} + y\left(x^2 - z^2\right)\hat{\mathbf{y}} + z\left(y^2 - x^2\right)\hat{\mathbf{z}}\right\}}$ (again, <u>not</u> unique).

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2) SOLUTION 1

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

for $f(r,\phi,z)$ we have

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}\right)\hat{x} + \left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y}\right)\hat{y}$$
$$+ \left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial z}\right)\hat{z}$$

$$r = \sqrt{x^2+y^2} \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = 0$$

$$\phi = \arctan(y/x) \Rightarrow \frac{\partial \phi}{\partial x} = \frac{1}{1+y^2/x^2}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{1+y^2/x^2}\cdot\left(\frac{x}{x^2}\right) = \frac{x}{x^2+y^2} \quad \text{and} \quad \frac{\partial \phi}{\partial z} = 0$$

$$z = z \Rightarrow \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0 \quad \text{and} \quad \frac{\partial z}{\partial z} = 1$$

Substituting into $\vec{\nabla} f$ we have

$$\left[\begin{array}{l} \hat{x} = \cos\phi\,\hat{r} - \sin\phi\,\hat{\phi} \\ \hat{y} = \sin\phi\,\hat{r} + \cos\phi\,\hat{\phi} \\ \hat{z} = \hat{z} \end{array}\right]$$

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial r}\frac{x}{r} - \frac{\partial f}{\partial \phi}\frac{y}{r^2}\right)\left(\cos\phi\,\hat{r} - \sin\phi\,\hat{\phi}\right) +$$
$$\left(\frac{\partial f}{\partial r}\frac{y}{r} + \frac{\partial f}{\partial \phi}\frac{x}{r^2}\right)\left(\sin\phi\,\hat{r} + \cos\phi\,\hat{\phi}\right) + \frac{\partial f}{\partial z}\hat{z}$$

using the fact that $x = r\cos\phi$ and $y = r\sin\phi$

$$\vec{\nabla} f = \hat{r}\left(\frac{\partial f}{\partial r}\cos^2\phi - \frac{\partial f}{\partial \phi}\frac{\sin\phi\cos\phi}{r} + \frac{\partial f}{\partial r}\sin^2\phi + \frac{\partial f}{\partial \phi}\frac{\sin\phi\cos\phi}{r}\right) +$$

$$\hat{\phi}\left(-\frac{\partial f}{\partial r}\sin\phi\cos\phi + \frac{\partial f}{\partial \phi}\frac{\sin^2\phi}{r} + \frac{\partial f}{\partial r}\sin\phi\cos\phi + \frac{\partial f}{\partial \phi}\frac{\cos^2\phi}{r}\right) +$$

$$\hat{z}\frac{\partial f}{\partial z}$$

Thus, $\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$

2) SOLUTION 2

$$d\vec{r} = \hat{r}\,dr + \hat{\phi}\,r\,d\phi + \hat{z}\,dz$$

by definition, $df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \phi}d\phi + \frac{\partial f}{\partial z}dz$

and $df = \vec{\nabla}f \cdot d\vec{r} = (\vec{\nabla}f)_r\,dr + (\vec{\nabla}f)_\phi\,d\phi + (\vec{\nabla}f)_z\,dz$

So, $\frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \phi}d\phi + \frac{\partial f}{\partial z}dz = (\vec{\nabla}f)_r\,dr + (\vec{\nabla}f)_\phi\,d\phi + (\vec{\nabla}f)_z\,dz$

Thus, $(\vec{\nabla}f)_r = \frac{\partial f}{\partial r}$, $(\vec{\nabla}f)_\phi = \frac{1}{r}\frac{\partial f}{\partial \phi}$, $(\vec{\nabla}f)_z = \frac{\partial f}{\partial z}$

i.e., $\vec{\nabla}f(r,\phi,z) = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$