~~The term in curly brackets is:~~

$$\left.\begin{cases} n=1 & : \ 1-1-2\cos(\pi/2)=0, \\ n=2 & : \ 1+1-2\cos(\pi)=4, \\ n=3 & : \ 1-1-2\cos(3\pi/2)=0, \\ n=4 & : \ 1+1-2\cos(2\pi)=0, \end{cases}\right\} \text{etc. (Zero if } n \text{ is odd or divisible by 4, otherwise 4.)}$$

~~Therefore~~

$$C_n = \begin{cases} 8V_0/n\pi, & n=2,6,10,14,\text{etc. (in general, } 4j+2, \text{ for } j=0,1,2,\ldots), \\ 0, & \text{otherwise.} \end{cases}$$

~~So~~

$$\boxed{V(x,y) = \frac{8V_0}{\pi}\sum_{n=2,6,10,\ldots}\frac{e^{-n\pi x/a}\sin(n\pi y/a)}{n}} = \frac{8V_0}{\pi}\sum_{j=0}^{\infty}\frac{e^{-(4j+2)\pi x/a}\sin[(4j+2)\pi y/a]}{(4j+2)}.$$

Problem 3.13

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots}\frac{1}{n}e^{-n\pi x/a}\sin(n\pi y/a) \quad \text{(Eq. 3.36)}; \quad \sigma = -\epsilon_0\frac{\partial V}{\partial n} \quad \text{(Eq. 2.49)}.$$

So

$$\begin{aligned} \sigma(y) &= -\epsilon_0\frac{\partial}{\partial x}\left\{\frac{4V_0}{\pi}\sum\frac{1}{n}e^{-n\pi x/a}\sin(n\pi y/a)\right\}\Bigg|_{x=0} = -\epsilon_0\frac{4V_0}{\pi}\sum\frac{1}{n}\left(-\frac{n\pi}{a}\right)e^{-n\pi x/a}\sin(n\pi y/a)\Bigg|_{x=0} \\ &= \boxed{\frac{4\epsilon_0 V_0}{a}\sum_{n=1,3,5,\ldots}\sin(n\pi y/a).} \end{aligned}$$

Or, using the closed form 3.37:

$$\begin{aligned} V(x,y) &= \frac{2V_0}{\pi}\tan^{-1}\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right) \Rightarrow \sigma = -\epsilon_0\frac{2V_0}{\pi}\frac{1}{1+\frac{\sin^2(\pi y/a)}{\sinh^2(\pi x/a)}}\left(\frac{-\sin(\pi y/a)}{\sinh^2(\pi x/a)}\right)\frac{\pi}{a}\cosh(\pi x/a)\Bigg|_{x=0} \\ &= \frac{2\epsilon_0 V_0}{a}\frac{\sin(\pi y/a)\cosh(\pi x/a)}{\sin^2(\pi y/a)+\sinh^2(\pi x/a)}\Bigg|_{x=0} = \boxed{\frac{2\epsilon_0 V_0}{a}\frac{1}{\sin(\pi y/a)}.} \end{aligned}$$

Summation of series Eq. 3.36

$$V(x,y) = \frac{4V_0}{\pi}I, \text{ where } I \equiv \sum_{n=1,3,5,\ldots}\frac{1}{n}e^{-n\pi x/a}\sin(n\pi y/a).$$

Now $\sin w = \mathcal{I}m\left(e^{iw}\right)$, so

$$I = \mathcal{I}m\sum\frac{1}{n}e^{-n\pi x/a}e^{in\pi y/a} = \mathcal{I}m\sum\frac{1}{n}\mathcal{Z}^n,$$

where $\mathcal{Z} \equiv e^{-\pi(x-iy)/a}$. Now

$$\begin{aligned} \sum_{1,3,5,\ldots}\frac{1}{n}\mathcal{Z}^n &= \sum_{j=0}^{\infty}\frac{1}{(2j+1)}\mathcal{Z}^{(2j+1)} = \int_0^{\mathcal{Z}}\left\{\sum_{j=0}^{\infty}u^{2j}\right\}du \\ &= \int_0^{\mathcal{Z}}\frac{1}{1-u^2}\,du = \frac{1}{2}\ln\left(\frac{1+\mathcal{Z}}{1-\mathcal{Z}}\right) = \frac{1}{2}\ln\left(Re^{i\theta}\right) = \frac{1}{2}\left(\ln R + i\,\theta\right), \end{aligned}$$

where $Re^{i\theta} = \frac{1+\mathcal{Z}}{1-\mathcal{Z}}$. Therefore

$$\begin{aligned} I &= \mathcal{I}m\left\{\frac{1}{2}(\ln R + i\,\theta)\right\} = \frac{1}{2}\theta. \quad \text{But } \frac{1+\mathcal{Z}}{1-\mathcal{Z}} = \frac{1+e^{-\pi(x-iy)/a}}{1-e^{-\pi(x-iy)/a}} = \frac{\left(1+e^{-\pi(x-iy)/a}\right)\left(1-e^{-\pi(x+iy)/a}\right)}{\left(1-e^{-\pi(x-iy)/a}\right)\left(1-e^{-\pi(x+iy)/a}\right)} \\ &= \frac{1+e^{-\pi x/a}\left(e^{i\pi y/a}-e^{-i\pi y/a}\right)-e^{-2\pi x/a}}{\left|1-e^{-\pi(x-iy)/a}\right|^2} = \frac{1+2ie^{-\pi x/a}\sin(\pi y/a)-e^{-2\pi x/a}}{\left|1-e^{-\pi(x-iy)/a}\right|^2}, \end{aligned}$$

so

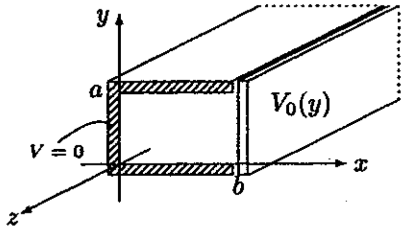
$$\tan\theta = \frac{2e^{-\pi x/a}\sin(\pi y/a)}{1-e^{-2\pi x/a}} = \frac{2\sin(\pi y/a)}{e^{\pi x/a}-e^{-\pi x/a}} = \frac{\sin(\pi y/a)}{\sinh(\pi x/a)}.$$

Therefore

$$I = \frac{1}{2}\tan^{-1}\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right), \text{ and } \boxed{V(x,y) = \frac{2V_0}{\pi}\tan^{-1}\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right).}$$

Problem 3.14

(a) $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$, with boundary conditions



$$\left\{\begin{array}{ll} \text{(i)} & V(x,0) = 0, \\ \text{(ii)} & V(x,a) = 0, \\ \text{(iii)} & V(0,y) = 0, \\ \text{(iv)} & V(b,y) = V_0(y). \end{array}\right\}$$

As in Ex. 3.4, separation of variables yields

$$V(x,y) = \left(Ae^{kx} + Be^{-kx}\right)(C\sin ky + D\cos ky).$$

Here (i)⇒ $D = 0$, (iii)⇒ $B = -A$, (ii)⇒ ka is an integer multiple of π:

$$V(x,y) = AC\left(e^{n\pi x/a} - e^{-n\pi x/a}\right)\sin(n\pi y/a) = (2AC)\sinh(n\pi x/a)\sin(n\pi y/a).$$

But $(2AC)$ is a constant, and the most general linear combination of separable solutions consistent with (i), (ii), (iii) is

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} C_n \sinh(n\pi x/a)\sin(n\pi y/a).}$$

It remains to determine the coefficients C_n so as to fit boundary condition (iv):

$$\sum C_n \sinh(n\pi b/a)\sin(n\pi y/a) = V_0(y). \text{ Fourier's trick} \Rightarrow C_n\sinh(n\pi b/a) = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy.$$

Therefore

$$\boxed{C_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a V_0(y)\sin(n\pi y/a)\,dy.}$$

(b) $C_n = \dfrac{2}{a\sinh(n\pi b/a)} V_0 \displaystyle\int_0^a \sin(n\pi y/a)\,dy = \dfrac{2V_0}{a\sinh(n\pi b/a)} \times \begin{Bmatrix} 0, & \text{if } n \text{ is even,} \\ \frac{2a}{n\pi}, & \text{if } n \text{ is odd.} \end{Bmatrix}$

$$\boxed{V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\dots} \frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi b/a)}.}$$

Problem 3.15

Same format as Ex. 3.5, only the boundary conditions are:

$$\begin{Bmatrix} \text{(i)} & V = 0 & \text{when} & x = 0, \\ \text{(ii)} & V = 0 & \text{when} & x = a, \\ \text{(iii)} & V = 0 & \text{when} & y = 0, \\ \text{(iv)} & V = 0 & \text{when} & y = a, \\ \text{(v)} & V = 0 & \text{when} & z = 0, \\ \text{(vi)} & V = V_0 & \text{when} & z = a. \end{Bmatrix}$$

This time we want sinusoidal functions in x and y, exponential in z:

$$X(x) = A\sin(kx) + B\cos(kx), \quad Y(y) = C\sin(ly) + D\cos(ly), \quad Z(z) = Ee^{\sqrt{k^2+l^2}z} + Ge^{-\sqrt{k^2+l^2}z}.$$

(i)⇒ $B = 0$; (ii)⇒ $k = n\pi/a$; (iii)⇒ $D = 0$; (iv)⇒ $l = m\pi/a$; (v)⇒ $E + G = 0$. Therefore

$$Z(z) = 2E\sinh(\pi\sqrt{n^2+m^2}z/a).$$

Putting this all together, and combining the constants, we have:

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\sin(n\pi x/a)\sin(m\pi y/a)\sinh(\pi\sqrt{n^2+m^2}z/a).$$

It remains to evaluate the constants $C_{n,m}$, by imposing boundary condition (vi):

$$V_0 = \sum\sum\left[C_{n,m}\sinh(\pi\sqrt{n^2+m^2})\right]\sin(n\pi x/a)\sin(m\pi y/a).$$

According to Eqs. 3.50 and 3.51:

$$C_{n,m}\sinh\left(\pi\sqrt{n^2+m^2}\right) = \left(\frac{2}{a}\right)^2 V_0 \int_0^a\int_0^a \sin(n\pi x/a)\sin(m\pi y/a)\,dx\,dy = \begin{Bmatrix} 0, & \text{if } n \text{ or } m \text{ is even,} \\ \dfrac{16V_0}{\pi^2 nm}, & \text{if both are odd.} \end{Bmatrix}$$

Therefore

$$\boxed{V(x,y,z) = \frac{16V_0}{\pi^2} \sum_{n=1,3,5,\dots}\ \sum_{m=1,3,5,\dots} \frac{1}{nm}\sin(n\pi x/a)\sin(m\pi y/a)\frac{\sinh\left(\pi\sqrt{n^2+m^2}z/a\right)}{\sinh\left(\pi\sqrt{n^2+m^2}\right)}.}$$

Problem 3.16

$$\begin{aligned} P_3(x) &= \frac{1}{8\cdot 6}\frac{d^3}{dx^3}\left(x^2-1\right)^3 = \frac{1}{48}\frac{d^2}{dx^2}3\left(x^2-1\right)^2 2x = \frac{1}{8}\frac{d^2}{dx^2}x\left(x^2-1\right)^2 \\ &= \frac{1}{8}\frac{d}{dx}\left[\left(x^2-1\right)^2 + 2x\left(x^2-1\right)2x\right] = \frac{1}{8}\frac{d}{dx}\left[\left(x^2-1\right)\left(x^2-1+4x^2\right)\right] \\ &= \frac{1}{8}\frac{d}{dx}\left[\left(x^2-1\right)\left(5x^2-1\right)\right] = \frac{1}{8}\left[2x\left(5x^2-1\right)+\left(x^2-1\right)10x\right] \\ &= \frac{1}{4}\left(5x^3 - x + 5x^3 - 5x\right) = \frac{1}{4}\left(10x^3-6x\right) = \boxed{\frac{5}{2}x^3 - \frac{3}{2}x.} \end{aligned}$$

We need to show that $P_3(\cos\theta)$ satisfies

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) = -l(l+1)P, \text{ with } l = 3,$$

where $P_3(\cos\theta) = \frac{1}{2}\cos\theta\left(5\cos^2\theta - 3\right)$.

$$\begin{aligned} \frac{dP_3}{d\theta} &= \frac{1}{2}\left[-\sin\theta\left(5\cos^2\theta-3\right) + \cos\theta(10\cos\theta(-\sin\theta)\right] = -\frac{1}{2}\sin\theta\left(5\cos^2\theta - 3 + 10\cos^2\theta\right) \\ &= -\frac{3}{2}\sin\theta\left(5\cos^2\theta-1\right). \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial\theta}\left(\sin\theta\frac{dP_3}{d\theta}\right) &= -\frac{3}{2}\frac{d}{d\theta}\left[\sin^2\theta\left(5\cos^2\theta-1\right)\right] = -\frac{3}{2}\left[2\sin\theta\cos\theta\left(5\cos^2\theta-1\right) + \sin^2\theta\left(-10\cos\theta\sin\theta\right)\right] \\ &= -3\sin\theta\cos\theta\left[5\cos^2\theta - 1 - 5\sin^2\theta\right]. \end{aligned}$$

$$\begin{aligned} \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) &= -3\cos\theta\left[5\cos^2 - 1 - 5\left(1-\cos^2\theta\right)\right] = -3\cos\theta\left(10\cos^2\theta - 6\right) \\ &= -3\cdot 4\cdot\frac{1}{2}\cos\theta\left(5\cos^2\theta-3\right) = -l(l+1)P_3. \quad \text{qed} \end{aligned}$$

$$\int_{-1}^{1} P_1(x)P_3(x)\,dx = \int_{-1}^{1}(x)\frac{1}{2}\left(5x^3-3x\right)\,dx = \frac{1}{2}\left(x^5-x^3\right)\Big|_{-1}^{1} = \frac{1}{2}(1-1+1-1) = 0. \checkmark$$

Problem 3.17

(a) *Inside:* $V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ (Eq. 3.66) where

$$A_l = \frac{(2l+1)}{2R^l}\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta \quad \text{(Eq. 3.69)}.$$

In this case $V_0(\theta) = V_0$ comes outside the integral, so

$$A_l = \frac{(2l+1)V_0}{2R^l}\int_0^\pi P_l(\cos\theta)\sin\theta\,d\theta.$$

But $P_0(\cos\theta) = 1$, so the integral can be written

$$\int_0^\pi P_0(\cos\theta)P_l(\cos\theta)\sin\theta\,d\theta = \left\{\begin{array}{ll} 0, & \text{if } l \neq 0 \\ 2, & \text{if } l = 0 \end{array}\right\} \quad \text{(Eq. 3.68).}$$

Therefore

$$A_l = \left\{\begin{array}{ll} 0, & \text{if } l \neq 0 \\ V_0, & \text{if } l = 0 \end{array}\right\}.$$

Plugging this into the general form:

$$V(r,\theta) = A_0\, r^0 P_0(\cos\theta) = \boxed{V_0.}$$

The potential is *constant throughout the sphere.*

Outside: $V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$ (Eq. 3.72), where

$$\begin{aligned} B_l &= \frac{(2l+1)}{2} R^{l+1} \int_0^\pi V_0(\theta) P_l(\cos\theta)\sin\theta\,d\theta \quad \text{(Eq. 3.73).} \\ &= \frac{(2l+1)}{2} R^{l+1} V_0 \int_0^\pi P_l(\cos\theta)\sin\theta\,d\theta = \left\{\begin{array}{ll} 0, & \text{if } l \neq 0 \\ RV_0, & \text{if } l = 0 \end{array}\right\}. \end{aligned}$$

Therefore $\boxed{V(r,\theta) = V_0 \frac{R}{r}}$ (i.e. equals V_0 at $r = R$, then falls off like $\frac{1}{r}$).

(b)

$$V(r,\theta) = \left\{\begin{array}{lll} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), & \text{for } r \le R & \text{(Eq. 3.78)} \\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), & \text{for } r \ge R & \text{(Eq. 3.79)} \end{array}\right\},$$

where

$$B_l = R^{2l+1} A_l \quad \text{(Eq. 3.81)}$$

and

$$\begin{aligned} A_l &= \frac{1}{2\epsilon_0 R^{l-1}} \int_0^\pi \sigma_0(\theta) P_l(\cos\theta)\sin\theta\,d\theta \quad \text{(Eq. 3.84)} \\ &= \frac{1}{2\epsilon_0 R^{l-1}} \sigma_0 \int_0^\pi P_l(\cos\theta)\sin\theta\,d\theta = \left\{\begin{array}{ll} 0, & \text{if } l \neq 0 \\ R\sigma_0/\epsilon_0, & \text{if } l = 0 \end{array}\right\}. \end{aligned}$$

Therefore

$$\boxed{V(r,\theta) = \left\{\begin{array}{ll} \dfrac{R\sigma_0}{\epsilon_0}, & \text{for } r \le R \\ \dfrac{R^2\sigma_0}{\epsilon_0}\dfrac{1}{r}, & \text{for } r \ge R \end{array}\right\}}.$$

Note: in terms of the total charge $Q = 4\pi R^2 \sigma_0$,

$$V(r,\theta) = \left\{ \begin{array}{ll} \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{R}, & \text{for } r \le R \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r}, & \text{for } r \ge R \end{array} \right\}$$

Problem 3.18

$$V_0(\theta) = k\cos(3\theta) = k\left[4\cos^3\theta - 3\cos\theta\right] = k\left[\alpha P_3(\cos\theta) + \beta P_1(\cos\theta)\right].$$

(I know that any 3rd order polynomial can be expressed as a linear combination of the first four Legendre polynomials; in this case, since the polynomial is *odd*, I only need P_1 and P_3.)

$$4\cos^3\theta - 3\cos\theta = \alpha\left[\frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right)\right] + \beta\cos\theta = \frac{5\alpha}{2}\cos^3\theta + \left(\beta - \frac{3}{2}\alpha\right)\cos\theta,$$

so

$$4 = \frac{5\alpha}{2} \Rightarrow \alpha = \frac{8}{5}; \quad -3 = \beta - \frac{3}{2}\alpha = \beta - \frac{3}{2}\cdot\frac{8}{5} = \beta - \frac{12}{5} \Rightarrow \beta = \frac{12}{5} - 3 = -\frac{3}{5}.$$

Therefore

$$V_0(\theta) = \frac{k}{5}\left[8P_3(\cos\theta) - 3P_1(\cos\theta)\right].$$

Now

$$V(r,\theta) = \left\{ \begin{array}{lll} \displaystyle\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), & \text{for } r \le R & \text{(Eq. 3.66)} \\ \displaystyle\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), & \text{for } r \ge R & \text{(Eq. 3.71)} \end{array} \right\},$$

where

$$\begin{aligned} A_l &= \frac{(2l+1)}{2R^l}\int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta \quad \text{(Eq. 3.69)} \\ &= \frac{(2l+1)}{2R^l}\frac{k}{5}\left\{8\int_0^{\pi} P_3(\cos\theta)P_l(\cos\theta)\sin\theta\,d\theta - 3\int_0^{\pi} P_1(\cos\theta)P_l(\cos\theta)\sin\theta\,d\theta\right\} \\ &= \frac{k}{5}\frac{(2l+1)}{2R^l}\left\{8\frac{2}{(2l+1)}\delta_{l3} - 3\frac{2}{(2l+1)}\delta_{l1}\right\} = \frac{k}{5}\frac{1}{R^l}\left[8\,\delta_{l3} - 3\,\delta_{l1}\right] \\ &= \left\{ \begin{array}{ll} 8k/5R^3, & \text{if } l = 3 \\ -3k/5R, & \text{if } l = 1 \end{array} \right\} \text{(zero otherwise).} \end{aligned}$$

Therefore

$$V(r,\theta) = -\frac{3k}{5R}rP_1(\cos\theta) + \frac{8k}{5R^3}r^3P_3(\cos\theta) = \boxed{\frac{k}{5}\left[8\left(\frac{r}{R}\right)^3 P_3(\cos\theta) - 3\left(\frac{r}{R}\right)P_1(\cos\theta)\right]},$$

or

$$\frac{k}{5}\left\{8\left(\frac{r}{R}\right)^3\frac{1}{2}\left[5\cos^3\theta - 3\cos\theta\right] - 3\left(\frac{r}{R}\right)\cos\theta\right\} \Rightarrow \boxed{V(r,\theta) = \frac{k}{5}\frac{r}{R}\cos\theta\left\{4\left(\frac{r}{R}\right)^2\left[5\cos^2\theta - 3\right] - 3\right\}}$$

~~(for $r \le R$). Meanwhile, $B_l = A_l R^{2l+1}$ (Eq. 3.81—this follows from the continuity of V at R). Therefore~~

$$B_l = \left\{ \begin{array}{ll} 8kR^4/5, & \text{if } l=3 \\ -3kR^2/5, & \text{if } l=1 \end{array} \right\} \quad \text{(zero otherwise).}$$

~~So~~

$$V(r,\theta) = \frac{-3kR^2}{5}\frac{1}{r^2}P_1(\cos\theta) + \frac{8kR^4}{5}\frac{1}{r^4}P_3(\cos\theta) = \boxed{\frac{k}{5}\left[8\left(\frac{R}{r}\right)^4 P_3(\cos\theta) - 3\left(\frac{R}{r}\right)^2 P_1(\cos\theta)\right]},$$

~~or~~

$$\boxed{V(r,\theta) = \frac{k}{5}\left(\frac{R}{r}\right)^2 \cos\theta \left\{4\left(\frac{R}{r}\right)^2 [5\cos^2\theta - 3] - 3\right\}}$$

~~(for $r \ge R$). Finally, using Eq. 3.83:~~

$$\begin{aligned}
\sigma(\theta) &= \epsilon_0 \sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l(\cos\theta) = \epsilon_0 \left[3A_1 P_1 + 7A_3 R^2 P_3\right] \\
&= \epsilon_0 \left[3\left(-\frac{3k}{5R}\right) P_1 + 7\left(\frac{8k}{5R^3}\right) R^2 P_3\right] = \boxed{\frac{\epsilon_0 k}{5R}\left[-9P_1(\cos\theta) + 56P_3(\cos\theta)\right]} \\
&= \frac{\epsilon_0 k}{5R}\left[-9\cos\theta + \frac{56}{2}\left(5\cos^3\theta - 3\cos\theta\right)\right] = \frac{\epsilon_0 k}{5R}\cos\theta[-9 + 28\cdot 5\cos^2\theta - 28\cdot 3] \\
&= \boxed{\frac{\epsilon_0 k}{5R}\cos\theta\left[140\cos^2\theta - 93\right].}
\end{aligned}$$

~~Problem 3.19~~

~~Use Eq. 3.83:~~ $\sigma(\theta) = \epsilon_0 \sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta)$. ~~But Eq. 3.69 says:~~ $A_l = \frac{2l+1}{2R^l}\int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta$.

~~Putting them together:~~

$$\sigma(\theta) = \frac{\epsilon_0}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta), \quad \text{with } C_l = \int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta. \quad \text{qed}$$

Problem 3.20

Set $V = 0$ on the equatorial plane, far from the sphere. Then the potential is the same as Ex. 3.8 *plus* the potential of a uniformly charged spherical shell:

$$\boxed{V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta + \frac{1}{4\pi\epsilon_0}\frac{Q}{r}.}$$

Problem 3.21

(a) $V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$ $(r > R)$, so $V(r,0) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(1) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right]$.

Since $r > R$ in this region, $\sqrt{r^2+R^2} = r\sqrt{1+(R/r)^2} = r\left[1 + \frac{1}{2}(R/r)^2 - \frac{1}{8}(R/r)^4 + \dots\right]$, so

$$\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} = \frac{\sigma}{2\epsilon_0} r\left[1 + \frac{1}{2}\frac{R^2}{r^2} - \frac{1}{8}\frac{R^4}{r^4} + \dots - 1\right] = \frac{\sigma}{2\epsilon_0}\left(\frac{R^2}{2r} - \frac{R^4}{8r^3} + \dots\right).$$

Comparing like powers of r, I see that $B_0 = \frac{\sigma R^2}{4\epsilon_0}$, $B_1 = 0$, $B_2 = -\frac{\sigma R^4}{16\epsilon_0}, \dots$. Therefore

$$\boxed{\begin{aligned} V(r,\theta) &= \frac{\sigma R^2}{4\epsilon_0}\left[\frac{1}{r} - \frac{R^2}{4r^3}P_2(\cos\theta) + \dots\right], \\ &= \frac{\sigma R^2}{4\epsilon_0 r}\left[1 - \frac{1}{8}\left(\frac{R}{r}\right)^2 (3\cos^2\theta - 1) + \dots\right], \end{aligned}} \quad \text{(for } r > R\text{).}$$

(b) $V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ $(r < R)$. In the northern hemispere, $0 \le \theta \le \pi/2$,

$$V(r,0) = \sum_{l=0}^{\infty} A_l r^l = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right].$$

Since $r < R$ in this region, $\sqrt{r^2+R^2} = R\sqrt{1+(r/R)^2} = R\left[1 + \frac{1}{2}(r/R)^2 - \frac{1}{8}(r/R)^4 + \dots\right]$. Therefore

$$\sum_{l=0}^{\infty} A_l r^l = \frac{\sigma}{2\epsilon_0}\left[R + \frac{1}{2}\frac{r^2}{R} - \frac{1}{8}\frac{r^4}{R^3} + \dots - r\right].$$

Comparing like powers: $A_0 = \frac{\sigma}{2\epsilon_0}R$, $A_1 = -\frac{\sigma}{2\epsilon_0}$, $A_2 = \frac{\sigma}{2\epsilon_0 R}, \dots$, so

$$\boxed{\begin{aligned} V(r,\theta) &= \frac{\sigma}{2\epsilon_0}\left[R - rP_1(\cos\theta) + \frac{1}{2R}P_2(\cos\theta) + \dots\right], \\ &= \frac{\sigma R}{2\epsilon_0}\left[1 - \left(\frac{r}{R}\right)\cos\theta + \frac{1}{4}\left(\frac{r}{R}\right)^2 (3\cos^2\theta - 1) + \dots\right], \end{aligned}} \quad \text{(for } r < R, \text{ northern hemisphere).}$$

In the southern hemisphere we'll have to go for $\theta = \pi$, using $P_l(-1) = (-1)^l$.

$$V(r,\pi) = \sum_{l=0}^{\infty} (-1)^l \bar{A}_l r^l = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right].$$

(I put an overbar on $\overline{A}_l$ to distinguish it from the northern A_l). The only difference is the sign of $\overline{A}_1$: $\overline{A}_1 = +(\sigma/2\epsilon_0)$, $\overline{A}_0 = A_0$, $\overline{A}_2 = A_2$. So:

$$\boxed{\begin{aligned} V(r,\theta) &= \frac{\sigma}{2\epsilon_0}\left[R + rP_1(\cos\theta) + \frac{1}{2R}r^2P_2(\cos\theta) + \dots\right], \\ &= \frac{\sigma R}{2\epsilon_0}\left[1 + \left(\frac{r}{R}\right)\cos\theta + \frac{1}{4}\left(\frac{r}{R}\right)^2(3\cos^2\theta - 1) + \dots\right], \end{aligned}}$$

(for $r < R$, southern hemisphere).

~~Problem 3.22~~

$$V(r,\theta) = \left\{ \begin{array}{ll} \sum\limits_{l=0}^{\infty} A_l r^l P_l(\cos\theta), & (r \le R) \text{ (Eq. 3.78)}, \\ \sum\limits_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), & (r \ge R) \text{ (Eq. 3.79)}, \end{array} \right\}$$

~~where~~ $B_l = A_l R^{2l+1}$ ~~(Eq. 3.81) and~~

$$\begin{aligned} A_l &= \frac{1}{2\epsilon_0 R^{l-1}} \int_0^{\pi} \sigma_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta \quad \text{(Eq. 3.84)} \\ &= \frac{1}{2\epsilon_0 R^{l-1}}\sigma_0 \left\{ \int_0^{\pi/2} P_l(\cos\theta)\sin\theta\, d\theta - \int_{\pi/2}^{\pi} P_l(\cos\theta)\sin\theta\, d\theta \right\} \quad (\text{let } x = \cos\theta) \\ &= \frac{\sigma_0}{2\epsilon_0 R^{l-1}} \left\{ \int_0^1 P_l(x)\, dx - \int_{-1}^0 P_l(x)\, dx \right\}. \end{aligned}$$

~~Now~~ $P_l(-x) = (-1)^l P_l(x)$, ~~since~~ $P_l(x)$ ~~is even, for even~~ l, ~~and odd, for odd~~ l. ~~Therefore~~

$$\int_{-1}^{0} P_l(x)\, dx = \int_1^0 P_l(-x)\, d(-x) = (-1)^l \int_0^1 P_l(x)\, dx,$$

~~and hence~~

$$A_l = \frac{\sigma_0}{2\epsilon_0 R^{l-1}}\left[1 - (-1)^l\right]\int_0^1 P_l(x)\, dx = \left\{ \begin{array}{ll} 0, & \text{if } l \text{ is even} \\ \frac{\sigma_0}{\epsilon_0 R^{l-1}} \int_0^1 P_l(x)\, dx, & \text{if } l \text{ is odd} \end{array} \right\}.$$

$$\sigma = -\epsilon_0 \left.\frac{\partial V}{\partial s}\right|_{s=R} = -\epsilon_0 E_0 \left(-\frac{R^2}{s^2} - 1\right)\cos\phi\bigg|_{s=R} = \boxed{2\epsilon_0 E_0 \cos\phi.}$$

Problem 3.25

~~*Inside:* $V(s,\phi) = a_0 + \sum_{k=1}^{\infty} s^k\left(a_k \cos k\phi + b_k \sin k\phi\right)$. (In this region $\ln s$ and s^{-k} are no good—they blow up at $s = 0$.)~~

~~*Outside:* $V(s,\phi) = \bar{a}_0 + \sum_{k=1}^{\infty} \frac{1}{s^k}\left(c_k \cos k\phi + d_k \sin k\phi\right)$. (Here $\ln s$ and s^k are no good at $s \to \infty$).~~

$$\sigma = -\epsilon_0 \left(\frac{\partial V_{\text{out}}}{\partial s} - \frac{\partial V_{\text{in}}}{\partial s}\right)\bigg|_{s=R} \qquad \text{(Eq. 2.36)}.$$

~~Thus~~

$$a \sin 5\phi = -\epsilon_0 \sum_{k=1}^{\infty}\left\{-\frac{k}{R^{k+1}}\left(c_k \cos k\phi + d_k \sin k\phi\right) - kR^{k-1}\left(a_k \cos k\phi + b_k \sin k\phi\right)\right\}.$$

~~Evidently $a_k = c_k = 0$; $b_k = d_k = 0$ except $k = 5$; $a = 5\epsilon_0\left(\frac{1}{R^6}d_5 + R^4 b_5\right)$. Also, V is continuous at $s = R$: $a_0 + R^5 b_5 \sin 5\phi = \bar{a}_0 + \frac{1}{R^5}d_5 \sin 5\phi$. So $a_0 = \bar{a}_0$ (might as well choose both zero); $R^5 b_5 = R^{-5} d_5$, or $d_5 = R^{10} b_5$. Combining these results: $a = 5\epsilon_0\left(R^4 b_5 + R^4 b_5\right) = 10\epsilon_0 R^4 b_5$; $b_5 = \frac{a}{10\epsilon_0 R^4}$; $d_5 = \frac{aR^6}{10\epsilon_0}$. Therefore~~

$$\boxed{V(s,\phi) = \frac{a \sin 5\phi}{10\epsilon_0}\left\{\begin{array}{ll} s^5/R^4, & \text{for } s < R, \\ R^6/s^5, & \text{for } s > R. \end{array}\right\}}$$

Problem 3.26

Monopole term:

$$Q = \int \rho\, d\tau = kR \int \left[\frac{1}{r^2}(R - 2r)\sin\theta\right] r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

But the r integral is

$$\int_0^R (R - 2r)\, dr = \left(Rr - r^2\right)\Big|_0^R = R^2 - R^2 = 0. \quad \text{So } Q = 0.$$

Dipole term:

$$\int r\cos\theta\, \rho\, d\tau = kR \int (r\cos\theta)\left[\frac{1}{r^2}(R - 2r)\sin\theta\right] r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

But the θ integral is

$$\int_0^\pi \sin^2\theta \cos\theta\, d\theta = \frac{\sin^3\theta}{3}\bigg|_0^\pi = \frac{1}{3}(0 - 0) = 0.$$

So the dipole contribution is likewise zero.

Quadrupole term:

$$\int r^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\rho\, d\tau = \frac{1}{2}kR \int\int r^2\left(3\cos^2\theta - 1\right)\left[\frac{1}{r^2}(R - 2r)\sin\theta\right] r^2 \sin\theta\, dr\, d\theta.$$

r integral:

$$\int_0^R r^2(R-2r)\,dr = \left(\frac{r^3}{3}R - \frac{r^4}{2}\right)\Big|_0^R = \frac{R^4}{3} - \frac{R^4}{2} = -\frac{R^4}{6}.$$

θ integral:

$$\int_0^\pi \underbrace{(3\cos^2\theta - 1)}_{3(1-\sin^2\theta)-1=2-3\sin^2\theta} \sin^2\theta\,d\theta = 2\int_0^\pi \sin^2\theta\,d\theta - 3\int_0^\pi \sin^4\theta\,d\theta$$

$$= 2\left(\frac{\pi}{2}\right) - 3\left(\frac{3\pi}{8}\right) = \pi\left(1-\frac{9}{8}\right) = -\frac{\pi}{8}.$$

ϕ integral:

$$\int_0^{2\pi} d\phi = 2\pi.$$

The whole integral is:

$$\frac{1}{2}kR\left(-\frac{R^4}{6}\right)\left(-\frac{\pi}{8}\right)(2\pi) = \frac{k\pi^2R^5}{48}.$$

For point P on the z axis ($r \to z$ in Eq. 3.95) the approximate potential is

$$\boxed{V(z) \cong \frac{1}{4\pi\epsilon_0}\frac{k\pi^2R^5}{48z^3}.} \quad \text{(Quadrupole.)}$$

Problem 3.27

$\mathbf{p} = (3qa - qa)\,\hat{\mathbf{z}} + (-2qa - 2q(-a))\,\hat{\mathbf{y}} = 2qa\,\hat{\mathbf{z}}$. Therefore

$$V \cong \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2},$$

and $\mathbf{p}\cdot\hat{\mathbf{r}} = 2qa\,\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 2qa\cos\theta$, so

$$V \cong \boxed{\frac{1}{4\pi\epsilon_0}\frac{2qa\cos\theta}{r^2}.} \quad \text{(Dipole.)}$$

Problem 3.28

(a) By symmetry, $\mathbf{p}$ is clearly in the z direction: $\mathbf{p} = p\,\hat{\mathbf{z}}$; $p = \int z\rho\,d\tau \Rightarrow \int z\sigma\,da$.

$$\begin{aligned} p &= \int (R\cos\theta)(k\cos\theta)R^3\sin\theta\,d\theta\,d\phi = 2\pi R^3 k\int_0^\pi \cos^2\theta\sin\theta\,d\theta = 2\pi R^3 k\left(-\frac{\cos^3\theta}{3}\right)\Big|_0^\pi \\ &= \frac{2}{3}\pi R^3 k[1-(-1)] = \frac{4\pi R^3 k}{3}; \quad \boxed{\mathbf{p} = \frac{4\pi R^3 k}{3}\,\hat{\mathbf{z}}.} \end{aligned}$$

(b)

$$V \cong \frac{1}{4\pi\epsilon_0}\frac{4\pi R^3 k}{3}\frac{\cos\theta}{r^2} = \boxed{\frac{kR^3}{3\epsilon_0}\frac{\cos\theta}{r^2}.} \quad \text{(Dipole.)}$$