Problem 2.39

Say the charge on the inner cylinder is Q, for a length L. The field is given by Gauss's law: $\int \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s\cdot L = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}Q \Rightarrow \mathbf{E} = \frac{Q}{2\pi\epsilon_0 L}\frac{1}{s}\hat{\mathbf{s}}$. Potential difference between the cylinders is

$$V(b)-V(a) = -\int_a^b \mathbf{E}\cdot d\mathbf{l} = -\frac{Q}{2\pi\epsilon_0 L}\int_a^b \frac{1}{s}ds = -\frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right).$$

As set up here, a is at the higher potential, so $V = V(a)-V(b) = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)$.

$C = \frac{Q}{V} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)}$, so capacitance *per unit length* is $\boxed{\dfrac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}}$.

Problem 2.40

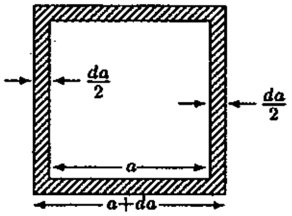
(a) W = (force)×(distance) = (pressure)×(area)×(distance) = $\boxed{\dfrac{\epsilon_0}{2}E^2 A\epsilon.}$

(b) W = (energy per unit volume)×(decrease in volume) = $\left(\epsilon_0\frac{E^2}{2}\right)(A\epsilon)$. Same as (a), confirming that the energy lost is equal to the work done.

Problem 2.41

From Prob. 2.4, the field at height z above the center of a square loop (side a) is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{4\lambda a z}{\left(z^2+\frac{a^2}{4}\right)\sqrt{z^2+\frac{a^2}{2}}}\,\hat{\mathbf{z}}.$$



Here $\lambda \to \sigma\frac{da}{2}$ (see figure), and we integrate over a from 0 to $\bar{a}$:

$$E = \frac{1}{4\pi\epsilon_0}2\sigma z\int_0^{\bar{a}}\frac{a\,da}{\left(z^2+\frac{a^2}{4}\right)\sqrt{z^2+\frac{a^2}{2}}}.\ \text{Let } u = \frac{a^2}{4},\text{ so } a\,da = 2\,du.$$

$$= \frac{1}{4\pi\epsilon_0}4\sigma z\int_0^{\bar{a}^2/4}\frac{du}{(u+z^2)\sqrt{2u+z^2}} = \frac{\sigma z}{\pi\epsilon_0}\left[\frac{2}{z}\tan^{-1}\left(\frac{\sqrt{2u+z^2}}{z}\right)\right]_0^{\bar{a}^2/4}$$

$$= \frac{2\sigma}{\pi\epsilon_0}\left\{\tan^{-1}\left(\frac{\sqrt{\frac{\bar{a}^2}{2}+z^2}}{z}\right)-\tan^{-1}(1)\right\};$$

$$\boxed{\mathbf{E} = \frac{2\sigma}{\pi\epsilon_0}\left[\tan^{-1}\sqrt{1+\frac{a^2}{2z^2}}-\frac{\pi}{4}\right]\hat{\mathbf{z}}.}$$

$a\to\infty$ (infinite plane): $E = \frac{2\sigma}{\pi\epsilon_0}\left[\tan^{-1}(\infty)-\frac{\pi}{4}\right] = \frac{2\sigma}{\pi\epsilon_0}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\sigma}{2\epsilon_0}$. ✓

$z\gg a$ (point charge): Let $f(x) = \tan^{-1}\sqrt{1+x}-\frac{\pi}{4}$, and expand as a Taylor series:

$$f(x) = f(0)+xf'(0)+\frac{1}{2}x^2f''(0)+\cdots$$

Here $f(0) = \tan^{-1}(1) - \frac{\pi}{4} = \frac{\pi}{4} - \frac{\pi}{4} = 0$; $f'(x) = \frac{1}{1+(1+x)}\frac{1}{2}\frac{1}{\sqrt{1+x}} = \frac{1}{2(2+x)\sqrt{1+x}}$, so $f'(0) = \frac{1}{4}$, so

$$f(x) = \frac{1}{4}x + (\ )x^2 + (\ )x^3 + \cdots$$

Thus (since $\frac{a^2}{2z^2} = x \ll 1$), $E \approx \frac{2\sigma}{\pi\epsilon_0}\left(\frac{1}{4}\frac{a^2}{2z^2}\right) = \frac{1}{4\pi\epsilon_0}\frac{\sigma a^2}{z^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}$. ✓

Problem 2.42

$$\rho = \epsilon_0 \nabla\cdot\mathbf{E} = \epsilon_0\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{A}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{B\sin\theta\cos\phi}{r}\right)\right\}$$
$$= \epsilon_0\left[\frac{1}{r^2}A + \frac{1}{r\sin\theta}\frac{B\sin\theta}{r}(-\sin\phi)\right] = \boxed{\frac{\epsilon_0}{r^2}(A - B\sin\phi).}$$

Problem 2.43

From Prob. 2.12, the field inside a uniformly charged sphere is: $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r}$. So the force per unit volume is $\mathbf{f} = \rho\mathbf{E} = \left(\frac{Q}{\frac{4}{3}\pi R^3}\right)\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)\mathbf{r} = \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2\mathbf{r}$, and the force in the z direction on $d\tau$ is:

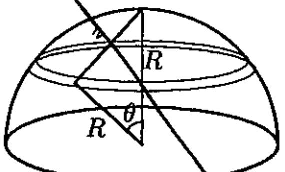
$$dF_z = f_z\,d\tau = \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 r\cos\theta(r^2\sin\theta\,dr\,d\theta\,d\phi).$$

The total force on the "northern" hemisphere is:

$$F_z = \int f_z\,d\tau = \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2\int_0^R r^3dr\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{2\pi}d\phi$$
$$= \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2\left(\frac{R^4}{4}\right)\left(\left.\frac{\sin^2\theta}{2}\right|_0^{\pi/2}\right)(2\pi) = \boxed{\frac{3Q^2}{64\pi\epsilon_0 R^2}.}$$

Problem 2.44

$$V_{\text{center}} = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma}{\mathscr{r}}da = \frac{1}{4\pi\epsilon_0}\frac{\sigma}{R}\int da = \frac{1}{4\pi\epsilon_0}\frac{\sigma}{R}(2\pi R^2) = \frac{\sigma R}{2\epsilon_0}$$



$$V_{\text{pole}} = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma}{\mathscr{r}}da\ \text{, with}\begin{cases}da = 2\pi R^2\sin\theta\,d\theta,\\ \mathscr{r}^2 = R^2 + R^2 - 2R^2\cos\theta = 2R^2(1-\cos\theta).\end{cases}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{\sigma(2\pi R^2)}{R\sqrt{2}}\int_0^{\pi/2}\frac{\sin\theta\,d\theta}{\sqrt{1-\cos\theta}} = \frac{\sigma R}{2\sqrt{2}\epsilon_0}\left.\left(2\sqrt{1-\cos\theta}\right)\right|_0^{\pi/2}$$
$$= \frac{\sigma R}{\sqrt{2}\epsilon_0}(1-0) = \frac{\sigma R}{\sqrt{2}\epsilon_0}. \qquad \therefore V_{\text{pole}} - V_{\text{center}} = \boxed{\frac{\sigma R}{2\epsilon_0}(\sqrt{2}-1).}$$

Problem 2.45

First let's determine the electric field inside and outside the sphere, using Gauss's law:

$$\epsilon_0\oint\mathbf{E}\cdot d\mathbf{a} = \epsilon_0 4\pi r^2 E = Q_{\text{enc}} = \int\rho\,d\tau = \int(k\bar{r})\bar{r}^2\sin\theta\,d\bar{r}\,d\theta\,d\phi = 4\pi k\int_0^r\bar{r}^3d\bar{r} = \begin{cases}\pi k r^4 & (r < R),\\ \pi k R^4 & (r > R).\end{cases}$$

So $\mathbf{E} = \frac{k}{4\epsilon_0} r^2 \,\hat{\mathbf{r}}$ $(r < R)$; $\quad \mathbf{E} = \frac{kR^4}{4\epsilon_0 r^2}\,\hat{\mathbf{r}}$ $(r > R)$.

Method I:

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau \text{ (Eq. 2.45)} = \frac{\epsilon_0}{2}\int_0^R \left(\frac{kr^2}{4\epsilon_0}\right)^2 4\pi r^2 dr + \frac{\epsilon_0}{2}\int_R^\infty \left(\frac{kR^4}{4\epsilon_0 r^2}\right)^2 4\pi r^2 dr$$

$$= 4\pi\frac{\epsilon_0}{2}\left(\frac{k}{4\epsilon_0}\right)^2 \left\{\int_0^R r^6 dr + R^8\int_R^\infty \frac{1}{r^2}dr\right\} = \frac{\pi k^2}{8\epsilon_0}\left\{\frac{R^7}{7} + R^8\left(-\frac{1}{r}\right)\Big|_R^\infty\right\} = \frac{\pi k^2}{8\epsilon_0}\left(\frac{R^7}{7} + R^7\right)$$

$$= \boxed{\frac{\pi k^2 R^7}{7\epsilon_0}.}$$

Method II:

$$W = \frac{1}{2}\int \rho V\, d\tau \quad \text{(Eq. 2.43).}$$

For $r < R$, $V(r) = -\int_\infty^r \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^R \left(\frac{kR^4}{4\epsilon_0 r^2}\right)dr - \int_R^r \left(\frac{kr^2}{4\epsilon_0}\right)dr = -\frac{k}{4\epsilon_0}\left\{R^4\left(-\frac{1}{r}\right)\Big|_\infty^R + \frac{r^3}{3}\Big|_R^r\right\}$

$$= -\frac{k}{4\epsilon_0}\left(-R^3 + \frac{r^3}{3} - \frac{R^3}{3}\right) = \frac{k}{3\epsilon_0}\left(R^3 - \frac{r^3}{4}\right).$$

$$\therefore W = \frac{1}{2}\int_0^R (kr)\left[\frac{k}{3\epsilon_0}\left(R^3 - \frac{r^3}{4}\right)\right] 4\pi r^2 dr = \frac{2\pi k^2}{3\epsilon_0}\int_0^R \left(R^3 r^3 - \frac{1}{4}r^6\right) dr$$

$$= \frac{2\pi k^2}{3\epsilon_0}\left\{R^3\frac{R^4}{4} - \frac{1}{4}\frac{R^7}{7}\right\} = \frac{\pi k^2 R^7}{2\cdot 3\epsilon_0}\left(\frac{6}{7}\right) = \frac{\pi k^2 R^7}{7\epsilon_0}. \checkmark$$

Problem 2.46

$$\mathbf{E} = -\nabla V = -A\frac{\partial}{\partial r}\left(\frac{e^{-\lambda r}}{r}\right)\hat{\mathbf{r}} = -A\left\{\frac{r(-\lambda)e^{-\lambda r} - e^{-\lambda r}}{r^2}\right\}\hat{\mathbf{r}} = \boxed{Ae^{-\lambda r}(1+\lambda r)\frac{\hat{\mathbf{r}}}{r^2}.}$$

$\rho = \epsilon_0\, \nabla\cdot\mathbf{E} = \epsilon_0 A\left\{e^{-\lambda r}(1+\lambda r)\,\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) + \frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla\left(e^{-\lambda r}(1+\lambda r)\right)\right\}$. But $\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^3(\mathbf{r})$ (Eq. 1.99), and $e^{-\lambda r}(1+\lambda r)\delta^3(\mathbf{r}) = \delta^3(\mathbf{r})$ (Eq. 1.88). Meanwhile,

$\nabla\left(e^{-\lambda r}(1+\lambda r)\right) = \hat{\mathbf{r}}\frac{\partial}{\partial r}\left(e^{-\lambda r}(1+\lambda r)\right) = \hat{\mathbf{r}}\left\{-\lambda e^{-\lambda r}(1+\lambda r) + e^{-\lambda r}\lambda\right\} = \hat{\mathbf{r}}(-\lambda^2 r e^{-\lambda r})$.

So $\frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla\left(e^{-\lambda r}(1+\lambda r)\right) = -\frac{\lambda^2}{r}e^{-\lambda r}$, and $\boxed{\rho = \epsilon_0 A\left[4\pi\delta^3(\mathbf{r}) - \frac{\lambda^2}{r}e^{-\lambda r}\right].}$

$$Q = \int \rho\, d\tau = \epsilon_0 A\left\{4\pi\int \delta^3(\mathbf{r})\,d\tau - \lambda^2\int \frac{e^{-\lambda r}}{r}4\pi r^2 dr\right\} = \epsilon_0 A\left(4\pi - \lambda^2 4\pi\int_0^\infty re^{-\lambda r}dr\right).$$

But $\int_0^\infty re^{-\lambda r}dr = \frac{1}{\lambda^2}$, so $Q = 4\pi\epsilon_0 A\left(1 - \frac{\lambda^2}{\lambda^2}\right) = \boxed{\text{zero.}}$

Problem 2.47

(a) Potential of $+\lambda$ is $V_+ = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_+}{a}\right)$, where s_+ is distance from λ_+ (Prob. 2.22).
Potential of $-\lambda$ is $V_- = +\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_-}{a}\right)$, where s_- is distance from λ_-.

Chapter 3

Special Techniques

Problem 3.1

The argument is exactly the same as in Sect. 3.1.4, except that since $z < R$, $\sqrt{z^2 + R^2 - 2zR} = (R - z)$, instead of $(z - R)$. Hence $V_{\text{ave}} = \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}[(z + R) - (R - z)] = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{R}}$. If there is more than one charge inside the sphere, the average potential due to interior charges is $\frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{R}$, and the average due to exterior charges is V_{center}, so $V_{\text{ave}} = V_{\text{center}} + \frac{Q_{\text{enc}}}{4\pi\epsilon_0 R}$. ✓

Problem 3.2

A stable equilibrium is a point of local minimum in the potential energy. Here the potential energy is qV. But we know that Laplace's equation allows no local minima for V. What *looks* like a minimum, in the figure, must in fact be a saddle point, and the box "leaks" through the center of each face.

Problem 3.3

Laplace's equation in *spherical* coordinates, for V dependent only on r, reads:

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0 \Rightarrow r^2\frac{dV}{dr} = c \text{ (constant)} \Rightarrow \frac{dV}{dr} = \frac{c}{r^2} \Rightarrow \boxed{V = -\frac{c}{r} + k.}$$

Example: potential of a uniformly charged sphere.

In *cylindrical* coordinates: $\nabla^2 V = \frac{1}{s}\frac{d}{ds}\left(s\frac{dV}{ds}\right) = 0 \Rightarrow s\frac{dV}{ds} = c \Rightarrow \frac{dV}{ds} = \frac{c}{s} \quad \Rightarrow \quad \boxed{V = c\ln s + k.}$

Example: potential of a long wire.

Problem 3.4

Same as proof of second uniqueness theorem, up to the equation $\oint_S V_3\mathbf{E}_3 \cdot d\mathbf{a} = -\int_{\mathcal{V}}(E_3)^2\,d\tau$. But on each surface, either $V_3 = 0$ (if V is specified on the surface), or else $E_{3\perp} = 0$ (if $\frac{\partial V}{\partial n} = -E_\perp$ is specified). So $\int_{\mathcal{V}}(E_3)^2 = 0$, and hence $\mathbf{E}_2 = \mathbf{E}_1$. qed

Problem 3.5

Putting $U = T = V_3$ into Green's identity:

$\int_{\mathcal{V}} [V_3\nabla^2 V_3 + \nabla V_3 \cdot \nabla V_3]\,d\tau = \oint_S V_3\nabla V_3 \cdot d\mathbf{a}$. But $\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{\rho}{\epsilon_0} + \frac{\rho}{\epsilon_0} = 0$, and $\nabla V_3 = -\mathbf{E}_3$. So $\int_{\mathcal{V}} E_3^2\,d\tau = -\oint_S V_2\mathbf{E}_3 \cdot d\mathbf{a}$, and the rest is the same as before.

Problem 3.6

Place image charges $+2q$ at $z=-d$ and $-q$ at $z=-3d$. Total force on $+q$ is

$$\mathbf{F}=\frac{q}{4\pi\epsilon_0}\left[\frac{-2q}{(2d)^2}+\frac{2q}{(4d)^2}+\frac{-q}{(6d)^2}\right]\hat{\mathbf{z}}=\frac{q^2}{4\pi\epsilon_0 d^2}\left(-\frac{1}{2}+\frac{1}{8}-\frac{1}{36}\right)\hat{\mathbf{z}}=\boxed{-\frac{1}{4\pi\epsilon_0}\left(\frac{29q^2}{72d^2}\right)\hat{\mathbf{z}}.}$$

Problem 3.7

(a) From Fig. 3.13: $\mathscr{r}=\sqrt{r^2+a^2-2ra\cos\theta}$; $\mathscr{r}'=\sqrt{r^2+b^2-2rb\cos\theta}$. Therefore:

$$\begin{aligned}\frac{q'}{\mathscr{r}'}&=-\frac{R}{a}\frac{q}{\sqrt{r^2+b^2-2rb\cos\theta}}\quad\text{(Eq. 3.15), while } b=\frac{R^2}{a}\text{ (Eq. 3.16).}\\&=-\frac{q}{\left(\frac{a}{R}\right)\sqrt{r^2+\frac{R^4}{a^2}-2r\frac{R^2}{a}\cos\theta}}=-\frac{q}{\sqrt{\left(\frac{ar}{R}\right)^2+R^2-2ra\cos\theta}}.\end{aligned}$$

Therefore:

$$V(r,\theta)=\frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}}+\frac{q'}{\mathscr{r}'}\right)=\boxed{\frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}}-\frac{1}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right\}.}$$

Clearly, when $r=R$, $V\to 0$.

(b) $\sigma=-\epsilon_0\frac{\partial V}{\partial n}$ (Eq. 2.49). In this case, $\frac{\partial V}{\partial n}=\frac{\partial V}{\partial r}$ at the point $r=R$. Therefore,

$$\begin{aligned}\sigma(\theta)&=-\epsilon_0\left(\frac{q}{4\pi\epsilon_0}\right)\left\{-\frac{1}{2}(r^2+a^2-2ra\cos\theta)^{-3/2}(2r-2a\cos\theta)\right.\\&\qquad\left.\left.+\frac{1}{2}\left(R^2+(ra/R)^2-2ra\cos\theta\right)^{-3/2}\left(\frac{a^2}{R^2}2r-2a\cos\theta\right)\right\}\right|_{r=R}\\&=-\frac{q}{4\pi}\left\{-(R^2+a^2-2Ra\cos\theta)^{-3/2}(R-a\cos\theta)+\left(R^2+a^2-2Ra\cos\theta\right)^{-3/2}\left(\frac{a^2}{R}-a\cos\theta\right)\right\}\\&=\frac{q}{4\pi}(R^2+a^2-2Ra\cos\theta)^{-3/2}\left[R-a\cos\theta-\frac{a^2}{R}+a\cos\theta\right]\\&=\boxed{\frac{q}{4\pi R}(R^2-a^2)(R^2+a^2-2Ra\cos\theta)^{-3/2}.}\\q_{\text{induced}}&=\int\sigma\,da=\frac{q}{4\pi R}(R^2-a^2)\int(R^2+a^2-2Ra\cos\theta)^{-3/2}R^2\sin\theta\,d\theta\,d\phi\\&=\frac{q}{4\pi R}(R^2-a^2)2\pi R^2\left[-\frac{1}{Ra}(R^2+a^2-2Ra\cos\theta)^{-1/2}\right]\Bigg|_0^\pi\\&=\frac{q}{2a}(a^2-R^2)\left[\frac{1}{\sqrt{R^2+a^2+2Ra}}-\frac{1}{\sqrt{R^2+a^2-2Ra}}\right].\end{aligned}$$

But $a>R$ (else q would be *inside*), so $\sqrt{R^2+a^2-2Ra}=a-R$.

$$\begin{aligned}&=\frac{q}{2a}(a^2-R^2)\left[\frac{1}{(a+R)}-\frac{1}{(a-R)}\right]=\frac{q}{2a}[(a-R)-(a+R)]=\frac{q}{2a}(-2R)\\&=\boxed{-\frac{qR}{a}=q'.}\end{aligned}$$

(c) The force on q, due to the sphere, is the same as the force of the image charge q', to wit:

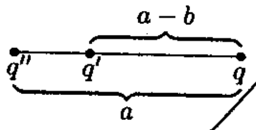
$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-b)^2} = \frac{1}{4\pi\epsilon_0}\left(-\frac{R}{a}q^2\right)\frac{1}{(a-R^2/a)^2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2Ra}{(a^2-R^2)^2}.$$

To bring q in from infinity to a, then, we do work

$$W = \frac{q^2R}{4\pi\epsilon_0}\int_{\infty}^{a}\frac{\bar{a}}{(\bar{a}^2-R^2)^2}\,d\bar{a} = \frac{q^2R}{4\pi\epsilon_0}\left[-\frac{1}{2}\frac{1}{(\bar{a}^2-R^2)}\right]\Bigg|_{\infty}^{a} = \boxed{-\frac{1}{4\pi\epsilon_0}\frac{q^2R}{2(a^2-R^2)}}.$$

Problem 3.8

Place a second image charge, q'', at the *center* of the sphere; this will not alter the fact that the sphere is an *equipotential*, but merely *increase* that potential from zero to $V_0 = \frac{1}{4\pi\epsilon_0}\frac{q''}{R}$;



$\boxed{q'' = 4\pi\epsilon_0 V_0 R \text{ at center of sphere.}}$

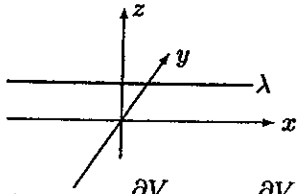
For a *neutral* sphere, $q' + q'' = 0$.

$$\begin{aligned}
F &= \frac{1}{4\pi\epsilon_0}q\left(\frac{q''}{a^2} + \frac{q'}{(a-b)^2}\right) = \frac{qq'}{4\pi\epsilon_0}\left(-\frac{1}{a^2} + \frac{1}{(a-b)^2}\right)\\
&= \frac{qq'}{4\pi\epsilon_0}\frac{b(2a-b)}{a^2(a-b)^2} = \frac{q(-Rq/a)}{4\pi\epsilon_0}\frac{(R^2/a)(2a-R^2/a)}{a^2(a-R^2/a)^2}\\
&= \boxed{-\frac{q^2}{4\pi\epsilon_0}\left(\frac{R}{a}\right)^3\frac{(2a^2-R^2)}{(a^2-R^2)^2}}.
\end{aligned}$$

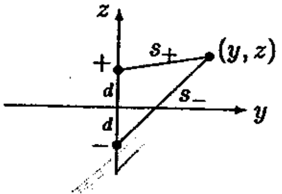
(Drop the minus sign, because the problem asks for the force of *attraction*.)

Problem 3.9

(a) Image problem: λ above, $-\lambda$ below. Potential was found in Prob. 2.47:



$$\begin{aligned}
V(y,z) &= \frac{2\lambda}{4\pi\epsilon_0}\ln(s_-/s_+) = \frac{\lambda}{4\pi\epsilon_0}\ln(s_-^2/s_+^2)\\
&= \boxed{\frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right\}}
\end{aligned}$$



(b) $\sigma = -\epsilon_0\frac{\partial V}{\partial n}$. Here $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial z}$, evaluated at $z=0$.

$$\begin{aligned}
\sigma(y) &= -\epsilon_0\frac{\lambda}{4\pi\epsilon_0}\left\{\frac{1}{y^2+(z+d)^2}2(z+d) - \frac{1}{y^2+(z-d)^2}2(z-d)\right\}\Bigg|_{z=0}\\
&= -\frac{2\lambda}{4\pi}\left\{\frac{d}{y^2+d^2} - \frac{-d}{y^2+d^2}\right\} = \boxed{-\frac{\lambda d}{\pi(y^2+d^2)}}.
\end{aligned}$$

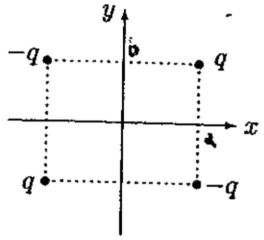
Check: Total charge induced on a strip of width l parallel to the y axis:

$$\begin{aligned}
q_{\text{ind}} &= -\frac{l\lambda d}{\pi}\int_{-\infty}^{\infty}\frac{1}{y^2+d^2}\,dy = -\frac{l\lambda d}{\pi}\left[\frac{1}{d}\tan^{-1}\left(\frac{y}{d}\right)\right]\Bigg|_{-\infty}^{\infty} = -\frac{l\lambda d}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]\\
&= -\lambda l.
\end{aligned}$$

Therefore $\lambda_{\text{ind}} = -\lambda$, as it should be.

Problem 3.10

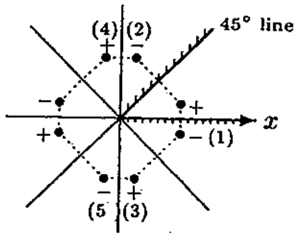
The image configuration is as shown.



$$\boxed{\begin{aligned} V(x,y) &= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{(x-a)^2+(y-b)^2+z^2}}+\frac{1}{\sqrt{(x+a)^2+(y+b)^2+z^2}}\right. \\ &\left.-\frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}}-\frac{1}{\sqrt{(x-a)^2+(y+b)^2+z^2}}\right\}.\end{aligned}}$$

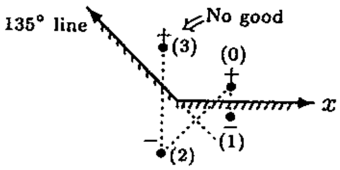
For this to work, $\boxed{\theta \text{ must be and integer divisor of } 180^\circ.}$ Thus 180°, 90°, 60°, 45°, etc., are OK, but no others. It works for 45°, say, with the charges as shown.

(Note the strategy: to make the x axis an equipotential ($V = 0$), you place the image charge (1) in the reflection point. To make the 45° line an equipotential, you place charge (2) at the image point. But that screws up the x axis, so you must now insert image (3) to balance (2). Moreover, to make the 45° line $V = 0$ you also need (4), to balance (1). But now, to restore the x axis to $V = 0$ you need (5) to balance (4), and so on.



why it works for $\theta = 45^\circ$

The reason this doesn't work for *arbitrary* angles is that you are eventually forced to place an image charge *within the original region of interest*, and that's not allowed—*all* images must go *outside* the region, or you're no longer dealing with the same problem at all.)



why it *doesn't* work for $\theta = 135^\circ$

Problem 3.11

From Prob. 2.47 (with $y_0 \to d$): $\boxed{V = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right],}$ where $a^2 = {y_0}^2 - R^2 \Rightarrow \boxed{a = \sqrt{d^2-R^2},}$

and

$$\left\{\begin{array}{l} a\coth(2\pi\epsilon_0 V_0/\lambda) = d \\ a\,\mathrm{csch}(2\pi\epsilon_0 V_0/\lambda) = R \end{array}\right\} \Rightarrow \text{(dividing)} \quad \frac{d}{R} = \cosh\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right), \text{ or } \boxed{\lambda = \frac{2\pi\epsilon_0 V_0}{\cosh^{-1}(d/R)}.}$$

Problem 3.12

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a}\sin(n\pi y/a) \quad \text{(Eq. 3.30)}, \quad \text{where } C_n = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy \quad \text{(Eq. 3.34)}.$$

In this case $V_0(y) = \left\{\begin{array}{ll} +V_0, & \text{for } 0 < y < a/2 \\ -V_0, & \text{for } a/2 < y < a \end{array}\right\}$ Therefore,

$$\begin{aligned} C_n &= \frac{2}{a}V_0\left\{\int_0^{a/2}\sin(n\pi y/a)\,dy - \int_{a/2}^{a}\sin(n\pi y/a)\,dy\right\} = \frac{2V_0}{a}\left\{-\left.\frac{\cos(n\pi y/a)}{(n\pi/a)}\right|_0^{a/2} + \left.\frac{\cos(n\pi y/a)}{(n\pi/a)}\right|_{a/2}^{a}\right\} \\ &= \frac{2V_0}{n\pi}\left\{-\cos\left(\frac{n\pi}{2}\right)+\cos(0)+\cos(n\pi)-\cos\left(\frac{n\pi}{2}\right)\right\} = \frac{2V_0}{n\pi}\left\{1+(-1)^n-2\cos\left(\frac{n\pi}{2}\right)\right\}.\end{aligned}$$