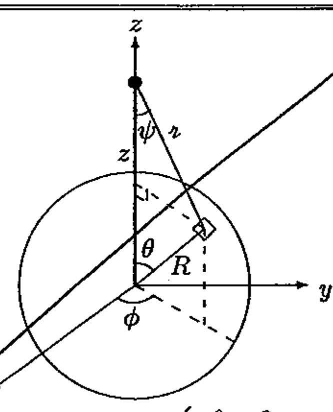
For $R \gg z$ the second term $\to 0$, so $\mathbf{E}_{\text{plane}} = \frac{1}{4\pi\epsilon_0}2\pi\sigma\hat{\mathbf{z}} = \boxed{\frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}.}$

For $z \gg R$, $\frac{1}{\sqrt{R^2+z^2}} = \frac{1}{z}\left(1+\frac{R^2}{z^2}\right)^{-1/2} \approx \frac{1}{z}\left(1-\frac{1}{2}\frac{R^2}{z^2}\right)$, so $[\,] \approx \frac{1}{z} - \frac{1}{z} + \frac{1}{2}\frac{R^2}{z^3} = \frac{R^2}{2z^3}$,

and $E = \frac{1}{4\pi\epsilon_0}\frac{2\pi R^2\sigma}{2z^2} = \frac{1}{4\pi\epsilon_0}\frac{Q}{z^2}$, where $Q = \pi R^2\sigma$. ✓

Problem 2.7



$\mathbf{E}$ is clearly in the z direction. From the diagram,

$dq = \sigma da = \sigma R^2 \sin\theta\, d\theta\, d\phi,$

$\imath^2 = R^2 + z^2 - 2Rz\cos\theta,$

$\cos\psi = \frac{z - R\cos\theta}{\imath}.$

So

$$E_z = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma R^2 \sin\theta\, d\theta\, d\phi (z - R\cos\theta)}{(R^2+z^2-2Rz\cos\theta)^{3/2}}. \qquad \int d\phi = 2\pi.$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_0^\pi \frac{(z-R\cos\theta)\sin\theta}{(R^2+z^2-2Rz\cos\theta)^{3/2}}\,d\theta. \qquad \text{Let } u = \cos\theta;\ du = -\sin\theta\, d\theta; \begin{Bmatrix} \theta = 0 \Rightarrow u = +1 \\ \theta = \pi \Rightarrow u = -1 \end{Bmatrix}.$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_{-1}^{1} \frac{z - Ru}{(R^2+z^2-2Rzu)^{3/2}}\,du.$$ Integral can be done by partial fractions—or look it up.

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\left[\frac{1}{z^2}\frac{zu - R}{\sqrt{R^2+z^2-2Rzu}}\right]_{-1}^{1} = \frac{1}{4\pi\epsilon_0}\frac{2\pi R^2\sigma}{z^2}\left\{\frac{(z-R)}{|z-R|} - \frac{(-z-R)}{|z+R|}\right\}.$$

For $z > R$ (outside the sphere), $E_z = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^2\sigma}{z^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}$, so $\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}}.}$

For $z < R$ (inside), $E_z = 0$, so $\boxed{\mathbf{E} = 0.}$

Problem 2.8

According to Prob. 2.7, all shells *interior* to the point (i.e. at smaller r) contribute as though their charge were concentrated at the center, while all exterior shells contribute nothing. Therefore:

$$\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{int}}}{r^2}\hat{\mathbf{r}},$$

where Q_{int} is the total charge interior to the point. *Outside* the sphere, *all* the charge is interior, so

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}.}$$

Inside the sphere, only that fraction of the total which is interior to the point counts:

$$Q_{\text{int}} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}Q = \frac{r^3}{R^3}Q, \quad \text{so } \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{r^3}{R^3}Q\frac{1}{r^2}\hat{\mathbf{r}} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r}.}$$

HW #4

Problem 2.9

(a) $\rho = \epsilon_0\,\nabla\cdot\mathbf{E} = \epsilon_0\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot kr^3\right) = \epsilon_0\frac{1}{r^2}k(5r^4) = \boxed{5\epsilon_0 kr^2.}$

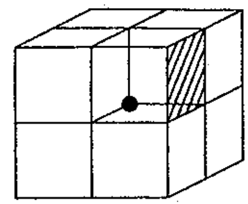
(b) *By Gauss's law:* $Q_{\text{enc}} = \epsilon_0 \oint \mathbf{E}\cdot d\mathbf{a} = \epsilon_0 (kR^3)(4\pi R^2) = \boxed{4\pi\epsilon_0 k R^5.}$

By direct integration: $Q_{\text{enc}} = \int \rho\, d\tau = \int_0^R (5\epsilon_0 k r^2)(4\pi r^2 dr) = 20\pi\epsilon_0 k \int_0^R r^4 dr = 4\pi\epsilon_0 k R^5. \checkmark$

Problem 2.10

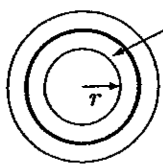
Think of this cube as one of 8 surrounding the charge. Each of the 24 squares which make up the surface of this larger cube gets the same flux as every other one, so:

$$\int\limits_{\substack{\text{one}\\\text{face}}} \mathbf{E}\cdot d\mathbf{a} = \frac{1}{24} \int\limits_{\substack{\text{whole}\\\text{large}\\\text{cube}}} \mathbf{E}\cdot d\mathbf{a}.$$



The latter is $\frac{1}{\epsilon_0}q$, by Gauss's law. Therefore $\boxed{\int\limits_{\substack{\text{one}\\\text{face}}} \mathbf{E}\cdot d\mathbf{a} = \frac{q}{24\epsilon_0}.}$

Problem 2.11

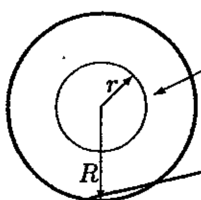


Gaussian surface: *Inside*: $\oint \mathbf{E}\cdot d\mathbf{a} = E(4\pi r^2) = \frac{1}{\epsilon_0} Q_{\text{enc}} = 0 \Rightarrow \boxed{\mathbf{E} = 0.}$

Gaussian surface: *Outside*: $E(4\pi r^2) = \frac{1}{\epsilon_0}(\sigma 4\pi R^2) \Rightarrow \boxed{\mathbf{E} = \frac{\sigma R^2}{\epsilon_0 r^2}\hat{\mathbf{r}}.}$

(As in Prob. 2.7.)

Problem 2.12



Gaussian surface

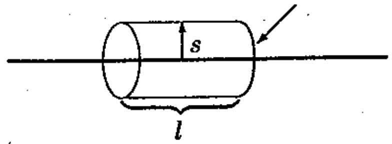
$\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 4\pi r^2 = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0}\frac{4}{3}\pi r^3 \rho.$ So

$$\boxed{\mathbf{E} = \frac{1}{3\epsilon_0}\rho r \hat{\mathbf{r}}.}$$

Since $Q_{\text{tot}} = \frac{4}{3}\pi R^3 \rho$, $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\hat{\mathbf{r}}$ (as in Prob. 2.8).

Problem 2.13

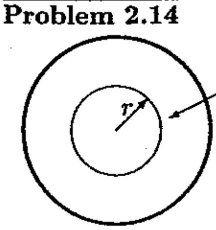
Gaussian surface



$\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s\cdot l = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0}\lambda l.$ So

$\boxed{\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}}$ (same as Ex. 2.1).

Problem 2.14



Gaussian surface $\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 4\pi r^2 = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0}\int \rho\, d\tau = \frac{1}{\epsilon_0}\int (k\bar{r})(\bar{r}^2 \sin\theta\, d\bar{r}\, d\theta\, d\phi)$

$$= \frac{1}{\epsilon_0} k\, 4\pi \int_0^r \bar{r}^3 d\bar{r} = \frac{4\pi k}{\epsilon_0}\frac{r^4}{4} = \frac{\pi k}{\epsilon_0} r^4.$$

$$\therefore \boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\pi k r^2 \hat{\mathbf{r}}.}$$

Problem 2.15

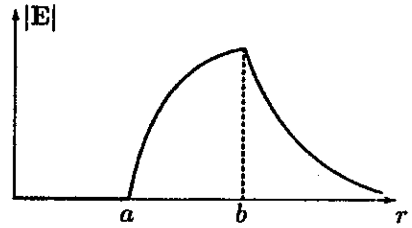
(i) $Q_{enc} = 0$, so $\boxed{\mathbf{E} = 0.}$

(ii) $\oint \mathbf{E}\cdot d\mathbf{a} = E(4\pi r^2) = \frac{1}{\epsilon_0}Q_{enc} = \frac{1}{\epsilon_0}\int \rho\, d\tau = \frac{1}{\epsilon_0}\int \frac{k}{\bar{r}^2}\bar{r}^2 \sin\theta\, d\bar{r}\, d\theta\, d\, phi$

$= \frac{4\pi k}{\epsilon_0}\int_a^r d\bar{r} = \frac{4\pi k}{\epsilon_0}(r-a) \therefore \boxed{\mathbf{E} = \frac{k}{\epsilon_0}\left(\frac{r-a}{r^2}\right)\hat{\mathbf{r}}.}$

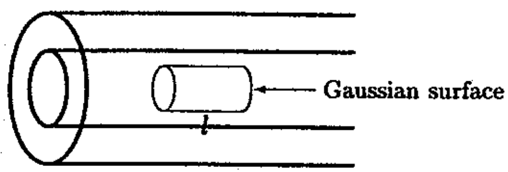
(iii) $E(4\pi r^2) = \frac{4\pi k}{\epsilon_0}\int_a^b d\bar{r} = \frac{4\pi k}{\epsilon_0}(b-a)$, so

$\boxed{\mathbf{E} = \frac{k}{\epsilon_0}\left(\frac{b-a}{r^2}\right)\hat{\mathbf{r}}.}$



Problem 2.16

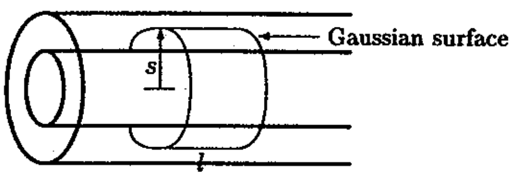
(i)



$\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s\cdot l = \frac{1}{\epsilon_0}Q_{enc} = \frac{1}{\epsilon_0}\rho\pi s^2 l;$

$\boxed{\mathbf{E} = \frac{\rho s}{2\epsilon_0}\hat{\mathbf{s}}.}$

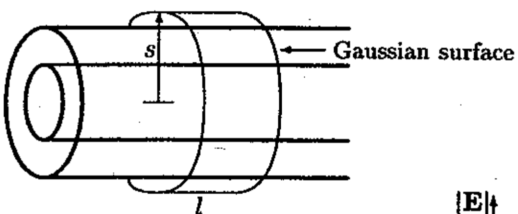
(ii)



$\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s\cdot l = \frac{1}{\epsilon_0}Q_{enc} = \frac{1}{\epsilon_0}\rho\pi a^2 l;$

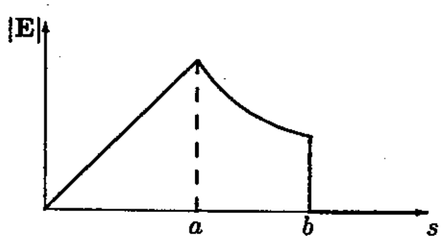
$\boxed{\mathbf{E} = \frac{\rho a^2}{2\epsilon_0 s}\hat{\mathbf{s}}.}$

(iii)



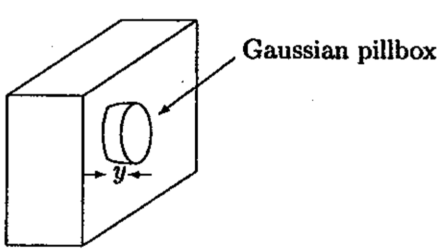
$\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s\cdot l = \frac{1}{\epsilon_0}Q_{enc} = 0;$

$\boxed{\mathbf{E} = 0.}$



Problem 2.17

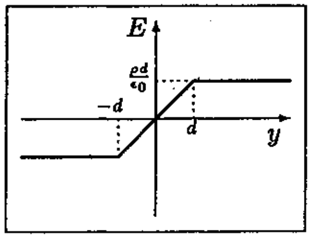
On the xz plane $E = 0$ by symmetry. Set up a Gaussian "pillbox" with one face in this plane and the other at y.



$\int \mathbf{E}\cdot d\mathbf{a} = E\cdot A = \frac{1}{\epsilon_0}Q_{enc} = \frac{1}{\epsilon_0}Ay\rho;$

$\boxed{\mathbf{E} = \frac{\rho}{\epsilon_0}y\,\hat{\mathbf{y}}}$ (for $|y| < d$).

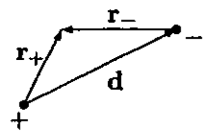
$Q_{enc} = \frac{1}{\epsilon_0} A d\rho \Rightarrow \boxed{\mathbf{E} = \frac{\rho}{\epsilon_0} d\,\hat{\mathbf{y}}}$ (for $y > d$).



Problem 2.18

From Prob. 2.12, the field inside the positive sphere is $\mathbf{E}_+ = \frac{\rho}{3\epsilon_0}\mathbf{r}_+$, where $\mathbf{r}_+$ is the vector from the positive center to the point in question. Likewise, the field of the negative sphere is $-\frac{\rho}{3\epsilon_0}\mathbf{r}_-$. So the *total* field is

$$\mathbf{E} = \frac{\rho}{3\epsilon_0}(\mathbf{r}_+ - \mathbf{r}_-)$$



But (see diagram) $\mathbf{r}_+ - \mathbf{r}_- = \mathbf{d}$. So $\boxed{\mathbf{E} = \frac{\rho}{3\epsilon_0}\mathbf{d}.}$

Problem 2.19

$$\nabla\times\mathbf{E} = \frac{1}{4\pi\epsilon_0}\nabla\times\int\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho\,d\tau = \frac{1}{4\pi\epsilon_0}\int\left[\nabla\times\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\right]\rho\,d\tau \quad \text{(since } \rho \text{ depends on } \mathbf{r}', \text{ not } \mathbf{r})$$

$$= 0 \quad \left(\text{since } \nabla\times\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 0, \text{ from Prob. 1.62}\right).$$

Problem 2.20

$$(1)\ \nabla\times\mathbf{E}_1 = k\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & 2yz & 3zx\end{vmatrix} = k\,[\hat{\mathbf{x}}(0-2y)+\hat{\mathbf{y}}(0-3z)+\hat{\mathbf{z}}(0-x)] \neq 0,$$

so $\mathbf{E}_1$ is an *impossible* electrostatic field.

$$(2)\ \nabla\times\mathbf{E}_2 = k\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2 & 2xy+z^2 & 2yz\end{vmatrix} = k\,[\hat{\mathbf{x}}(2z-2z)+\hat{\mathbf{y}}(0-0)+\hat{\mathbf{z}}(2y-2y)] = 0,$$

so $\mathbf{E}_2$ is a *possible* electrostatic field.

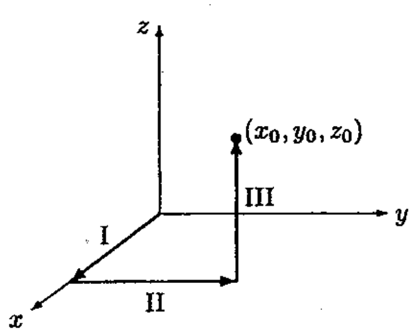
Let's go by the indicated path:

$$\mathbf{E}\cdot d\mathbf{l} = (y^2\,dx + (2xy+z^2)dy + 2yz\,dz)k$$

Step I: $y = z = 0$; $dy = dz = 0$. $\mathbf{E}\cdot d\mathbf{l} = ky^2\,dx = 0$.

Step II: $x = x_0$, $y: 0 \to y_0$, $z = 0$. $dx = dz = 0$.
$\mathbf{E}\cdot d\mathbf{l} = k(2xy+z^2)dy = 2kx_0y\,dy$.
$\int_{II}\mathbf{E}\cdot d\mathbf{l} = 2kx_0\int_0^{y_0} y\,dy = kx_0y_0^2$.

Step III: $x = x_0$, $y = y_0$, $z: 0 \to z_0$; $dx = dy = 0$.



$\mathbf{E}\cdot d\mathbf{l} = 2kyz\,dz = 2ky_0z\,dz.$
$\int_{III}\mathbf{E}\cdot d\mathbf{l} = 2y_0k\int_0^{z_0} z\,dz = ky_0z_0^2.$

$V(x_0,y_0,z_0) = -\int_0^{(x_0,y_0,z_0)}\mathbf{E}\cdot d\mathbf{l} = -k(x_0y_0^2+y_0z_0^2)$, or $\boxed{V(x,y,z) = -k(xy^2+yz^2).}$

Check: $-\nabla V = k\left[\frac{\partial}{\partial x}(xy^2+yz^2)\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}(xy^2+yz^2)\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}(xy^2+yz^2)\,\hat{\mathbf{z}}\right] = k[y^2\,\hat{\mathbf{x}} + (2xy+z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}] = \mathbf{E}.$ ✓

Problem 2.21

$V(r) = -\int_\infty^r \mathbf{E}\cdot d\mathbf{l}.$ $\begin{cases}\text{Outside the sphere } (r>R): & \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}.\\ \text{Inside the sphere } (r<R): & \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}.\end{cases}$

So for $r>R$: $V(r) = -\int_\infty^r\left(\frac{1}{4\pi\epsilon_0}\frac{q}{\bar r^2}\right)d\bar r = \frac{1}{4\pi\epsilon_0}q\left(\frac{1}{\bar r}\right)\Big|_\infty^r = \boxed{\frac{q}{4\pi\epsilon_0}\frac{1}{r},}$

and for $r<R$: $V(r) = -\int_\infty^R\left(\frac{1}{4\pi\epsilon_0}\frac{q}{\bar r^2}\right)d\bar r - \int_R^r\left(\frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\bar r\right)d\bar r = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R} - \frac{1}{R^3}\left(\frac{r^2-R^2}{2}\right)\right]$

$= \boxed{\frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(3-\frac{r^2}{R^2}\right).}$

When $r>R$, $\nabla V = \frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{\mathbf{r}} = -\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$, so $\mathbf{E} = -\nabla V = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}.$ ✓

When $r<R$, $\nabla V = \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\frac{\partial}{\partial r}\left(3-\frac{r^2}{R^2}\right)\hat{\mathbf{r}} = \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(-\frac{2r}{R^2}\right)\hat{\mathbf{r}} = -\frac{q}{4\pi\epsilon_0}\frac{r}{R^3}\hat{\mathbf{r}}$; so $\mathbf{E} = -\nabla V = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}.$ ✓

Problem 2.22

$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{s}\hat{\mathbf{s}}$ (Prob. 2.13). In this case we cannot set the reference point at ∞, since the charge itself extends to ∞. Let's set it at $s=a$. Then

$V(s) = -\int_a^s\left(\frac{1}{4\pi\epsilon_0}\frac{2\lambda}{\bar s}\right)d\bar s = \boxed{-\frac{1}{4\pi\epsilon_0}2\lambda\ln\left(\frac{s}{a}\right).}$

(In this form it is clear why $a=\infty$ would be no good—likewise the other "natural" point, $a=0$.)

$\nabla V = -\frac{1}{4\pi\epsilon_0}2\lambda\frac{\partial}{\partial s}\left(\ln\left(\frac{s}{a}\right)\right)\hat{\mathbf{s}} = -\frac{1}{4\pi\epsilon_0}2\lambda\frac{1}{s}\hat{\mathbf{s}} = -\mathbf{E}.$ ✓

Problem 2.23

$V(0) = -\int_\infty^0\mathbf{E}\cdot d\mathbf{l} = -\int_\infty^b\left(\frac{k}{\epsilon_0}\frac{(b-a)}{r^2}\right)dr - \int_b^a\left(\frac{k}{\epsilon_0}\frac{(r-a)}{r^2}\right)dr - \int_a^0(0)dr = \frac{k}{\epsilon_0}\frac{(b-a)}{b} - \frac{k}{\epsilon_0}\left(\ln\left(\frac{a}{b}\right) + a\left(\frac{1}{a}-\frac{1}{b}\right)\right)$

$= \frac{k}{\epsilon_0}\left\{1-\frac{a}{b}-\ln\left(\frac{a}{b}\right)-1+\frac{a}{b}\right\} = \boxed{\frac{k}{\epsilon_0}\ln\left(\frac{b}{a}\right).}$

Problem 2.24

Using Eq. 2.22 and the fields from Prob. 2.16:

$V(b)-V(0) = -\int_0^b\mathbf{E}\cdot d\mathbf{l} = -\int_0^a\mathbf{E}\cdot d\mathbf{l} - \int_a^b\mathbf{E}\cdot d\mathbf{l} = -\frac{\rho}{2\epsilon_0}\int_0^a s\,ds - \frac{\rho a^2}{2\epsilon_0}\int_a^b\frac{1}{s}ds$

$= -\left(\frac{\rho}{2\epsilon_0}\right)\frac{s^2}{2}\Big|_0^a + \frac{\rho a^2}{2\epsilon_0}\ln s\Big|_a^b = \boxed{-\frac{\rho a^2}{4\epsilon_0}\left(1+2\ln\left(\frac{b}{a}\right)\right).}$

Problem 2.25

(a) $\boxed{V = \frac{1}{4\pi\epsilon_0}\frac{2q}{\sqrt{z^2+\left(\frac{d}{2}\right)^2}}.}$