
Problem 1.39

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \tfrac{1}{r^2}\tfrac{\partial}{\partial r}(r^2\, r\cos\theta) + \tfrac{1}{r\sin\theta}\tfrac{\partial}{\partial\theta}(\sin\theta\, r\sin\theta) + \tfrac{1}{r\sin\theta}\tfrac{\partial}{\partial\phi}(r\sin\theta\cos\phi)\\ &= \tfrac{1}{r^2}\,3r^2\cos\theta + \tfrac{1}{r\sin\theta}\,r\,2\sin\theta\cos\theta + \tfrac{1}{r\sin\theta}\,r\sin\theta(-\sin\phi)\\ &= 3\cos\theta + 2\cos\theta - \sin\phi = 5\cos\theta - \sin\phi\end{aligned}$$

$$\int(\nabla\cdot\mathbf{v})d\tau = \int(5\cos\theta - \sin\phi)\, r^2\sin\theta\, dr\, d\theta\, d\phi = \int_0^R r^2\, dr \int_0^{\frac{\pi}{2}} \underbrace{\left[\int_0^{2\pi}(5\cos\theta - \sin\phi)\, d\phi\right]}_{\hookrightarrow 2\pi(5\cos\theta)} d\theta\sin\theta$$

$$= \left(\tfrac{R^3}{3}\right)(10\pi)\underbrace{\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\ , d\theta}_{\hookrightarrow \frac{\sin^2\theta}{2}\Big|_0^{\frac{\pi}{2}} = \frac{1}{2}}$$

$$= \boxed{\tfrac{5\pi}{3}R^3.}$$

Two surfaces—one the hemisphere: $d\mathbf{a} = R^2\sin\theta\, d\theta\, d\phi\,\hat{\mathbf{r}}$; $r = R$; $\phi : 0 \to 2\pi$, $\theta : 0 \to \frac{\pi}{2}$.

$\int\mathbf{v}\cdot d\mathbf{a} = \int(r\cos\theta)R^2\sin\theta\, d\theta\, d\phi = R^3\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\, d\theta\int_0^{2\pi}d\phi = R^3\left(\frac{1}{2}\right)(2\pi) = \pi R^3.$

other the flat bottom: $d\mathbf{a} = (dr)(r\sin\theta\, d\phi)(+\hat{\boldsymbol\theta}) = r\, dr\, d\phi\,\hat{\boldsymbol\theta}$ (here $\theta = \frac{\pi}{2}$). $r : 0 \to R$, $\phi : 0 \to 2\pi$.

$\int\mathbf{v}\cdot d\mathbf{a} = \int(r\sin\theta)(r\, dr\, d\phi) = \int_0^R r^2\, dr\int_0^{2\pi}d\phi = 2\pi\frac{R^3}{3}.$

Total: $\int\mathbf{v}\cdot d\mathbf{a} = \pi R^3 + \frac{2}{3}\pi R^3 = \frac{5}{3}\pi R^3$. ✓

Problem 1.44

(a) $\int_{-2}^{2}(2x+3)\frac{1}{3}\delta(x)\, dx = \frac{1}{3}(0+3) = \boxed{1.}$

(b) By Eq. 1.94, $\delta(1-x) = \delta(x-1)$, so $1 + 3 + 2 = \boxed{6.}$

(c) $\int_{-1}^{1} 9x^2\frac{1}{3}\,\delta(x+\frac{1}{3})\, dx = 9\left(-\frac{1}{3}\right)^2\frac{1}{3} = \boxed{\frac{1}{3}.}$

(d) $\boxed{1 \text{ (if } a > b\text{), } 0 \text{ (if } a < b\text{).}}$

Problem 1.46

(a) $\boxed{\rho(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{r}').}$ Check: $\int\rho(\mathbf{r})d\tau = q\int\delta^3(\mathbf{r} - \mathbf{r}')\, d\tau = q.$ ✓

(b) $\boxed{\cancel{\rho(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{r}') - q\delta^3(\mathbf{r}).}}$ $\rho(\vec{r}) = q\delta^3(\vec{r} - \vec{a}) - q\delta^3(\vec{r})$

(c) Evidently $\rho(r) = A\delta(r - R)$. To determine the constant A, we require

$Q = \int\rho\, d\tau = \int A\delta(r-R)4\pi r^2\, dr = A\,4\pi R^2.$ So $A = \frac{Q}{4\pi R^2}$. $\boxed{\rho(r) = \frac{Q}{4\pi R^2}\delta(r - R).}$

Problem 1.48

First method: use Eq. 1.99 to write $J = \int e^{-r}\left(4\pi\delta^3(\mathbf{r})\right)\, d\tau = 4\pi e^{-0} = \boxed{4\pi.}$

Second method: integrating by parts (use Eq. 1.59).

$$\begin{aligned} J &= -\int_{\mathcal{V}}\frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla(e^{-r})\, d\tau + \oint_{\mathcal{S}} e^{-r}\frac{\hat{\mathbf{r}}}{r^2}\cdot d\mathbf{a}.\quad \text{But}\quad \nabla\left(e^{-r}\right) = \left(\frac{\partial}{\partial r}e^{-r}\right)\hat{\mathbf{r}} = -e^{-r}\hat{\mathbf{r}}.\\ &= \int\frac{1}{r^2}e^{-r}4\pi r^2\, dr + \int e^{-r}\frac{\hat{\mathbf{r}}}{r^2}\cdot r^2\sin\theta\, d\theta\, d\phi\,\hat{\mathbf{r}} = 4\pi\int_0^{\overset{R}{\cancel{\infty}}} e^{-r}\, dr + e^{-R}\int\sin\theta\, d\theta\, d\phi\\ &= 4\pi\left(-e^{-r}\right)\Big|_0^{\overset{R}{\cancel{\infty}}} + 4\pi e^{-R} = \cancel{4\pi\left(-e^{-\infty} + e^{-0}\right)} = 4\pi. ✓\quad \text{(Here } R = \infty,\text{ so } e^{-R} = 0.)\end{aligned}$$

$\hookrightarrow -4\pi e^{-R} + 4\pi e^{-0} + 4\pi e^{-R} = 4\pi$