Coupled Oscillations

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A system with n degrees of freedom operating close to equilibrium has the equations of motion

$$M d^2\eta/dt^2 = -K\eta \tag{1}$$

where M is an n by n "mass" matrix derived from the expression for the kinetic energy, K is an n by n "spring constant" matrix derived from the expression for the potential energy, η is an n by 1 vector of generalized coordinates, and t is time. Both M and K are real, symmetric matrices so they may be diagonalized by similarity transformations using unitary matrices.

If the system had only one degree of freedom (so that M and K were just numbers) then the next step would be to divide by M. However, this does not work well for the general case. Instead two divisions by the square root of M are needed. Both $M^{1/2}$ and $M^{-1/2}$ may be obtained by invoking the similarity transformation that diagonalizes the symmetric matrix M and then taking the square roots (or inverse square roots) of the diagonal terms in the diagonal matrix. Note that both $M^{1/2}$ and $M^{-1/2}$ are symmetric matrices.

Using $M^{1/2}$ and $M^{-1/2}$, equation (1) may be modified to the form

$$M^{-1/2}M d^2\eta/dt^2 = -M^{-1/2}KM^{-1/2}M^{1/2}\eta \tag{2}$$

Then, introducing

$$\xi = M^{1/2}\eta \tag{3}$$

and

$$W = M^{-1/2}KM^{-1/2} \tag{4}$$

(also a symmetric matrix) allows equation (2) to be written as

$$d^2\xi/dt^2 = -W\xi \tag{5}$$

Now let the similarity transformation that diagonalizes W be performed by the unitary matrix U to produce the diagonal matrix D, i.e.

$$U^{T}WU = D \tag{6}$$

with

$$U^{-1} = U^{T} \tag{7}$$

Then, multiplying equation (5) by U^{T} and using equations (6) and (7) yields

$$U^{T}d^{2}\xi/dt^{2} \quad = -\,DU^{T}\xi \tag{8}$$

Finally introduce

$$\zeta = U^{T}\xi \tag{9}$$

and, from equation (8), arrive at

$$d^{2}\zeta/dt^{2} = -\,D\zeta \tag{10}$$

This is a set of n uncoupled equations (D is a diagonal matrix). Let

$$\omega_{k}^{2} = D_{kk} \tag{11}$$

then the components of ζ satisfy

$$d^{2}\zeta_{k} \ /dt^{2} = -\,\omega_{k}^{2}\zeta_{k} \tag{12}$$

with solutions

$$\zeta_{k} = \rho_{k}\cos(\omega_{k}t - \varphi_{k}) \tag{13}$$

where ρ_{k} and φ_{k} are constants of integration. These are the “normal modes” of the system. The original generalized coordinates may be recovered using equations (3) and (9) as

$$\eta = M^{-1/2}U\zeta \tag{14}$$

Thus, each generalized coordinate (component of η) is a linear combination of the normal modes (components of ζ).

The combination of matrices in equation (14)

$$A = M^{-1/2}U \tag{15}$$

is a congruence transformation and combines a rotation (U) with a scale change ($M^{-1/2}$). Note that $A^{-1} \neq A^{T}$ but, from equation (15), it follows that

$$A^{T}MA = 1 \tag{16}$$

where *1* is the identity matrix. Also, combining equations (4), (6), and (15) yields

$$A^{T}KA = D \tag{17}$$

Equations (16) and (17) further provide

$$KA = MAD \tag{18}$$

The original problem, equation (1), may now be solved by assuming harmonic behavior and using equations (16) and (18) directly. If a_k is the k^{th} column of the matrix A then equations (11) and (18) yield

$$(K - \omega_k^2 M)a_k = 0 \tag{19}$$

while equation (16) provides the normalization of the a_k's as

$$a_k^{T} M a_k = 1 \tag{20}$$

Hence the ω_k^2 are determined by

$$|K - \omega_k^2 M| = 0 \tag{21}$$

and then the a_k's are determined by equations (19) and (20).