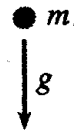


# Comprehensive Exam — Spring 1999

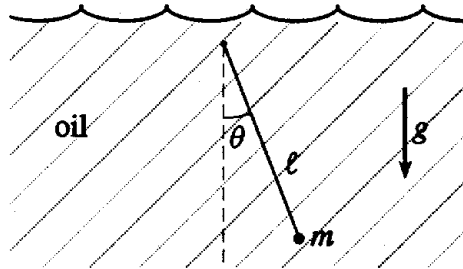
## Classical Mechanics

1. Consider a particle of mass  $m$  whose motion starts from rest in a constant gravitational field. If a resisting force proportional to the square of the velocity, i.e.,  $kmv^2$ , is encountered, calculate the distance  $s$  that the particle falls while accelerating from  $v_0$  to  $v_1$ . Express your result in terms of  $g$ ,  $k$ ,  $v_0$  and  $v_1$ .

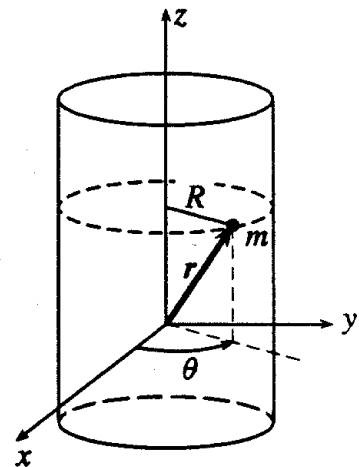


2. Given a pendulum of length  $\ell$  and a bob of mass  $m$  at its end moving through oil with  $\theta$  decreasing (see figure). The massive bob undergoes small oscillations, but the oil retards the bob's motion with a resistive force proportional to the speed with  $F_{\text{res}} = 2m\sqrt{g/\ell}(\ell\dot{\theta})$ . The bob is initially pulled back at  $t = 0$  with  $\theta = \alpha$  and  $\dot{\theta} = 0$ .

Find the angular displacement  $\theta$  and velocity  $\dot{\theta}$  as a function of time. Sketch the phase diagram for  $\sqrt{g/\ell} = 10 \text{ s}^{-1}$  and  $\alpha = 10^{-2} \text{ rad}$ .

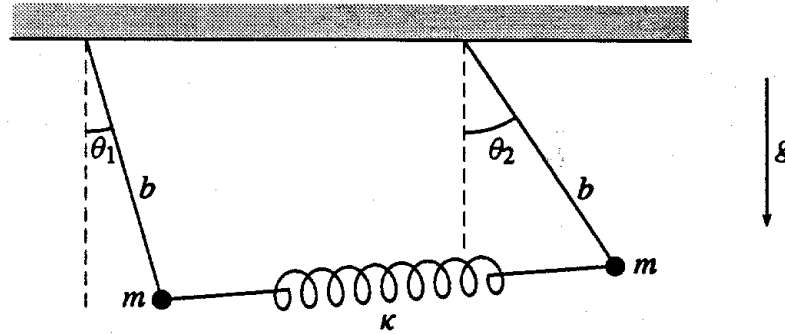


3. Use the Hamiltonian method to find the equation of motion of a particle of mass  $m$  constrained to move on the surface of a cylinder defined by  $x^2 + y^2 = R^2$ . The particle is subject to a force directed toward the origin and proportional to the distance of the particle from the origin:  $F = -\kappa r$ . Discuss conservation of momentum, angular momentum, and energy.



4. Determine the eigenfrequencies and describe the normal mode motion for two pendula of equal lengths  $b$  and equal masses  $m$ , connected by a spring of force constant  $\kappa$ , as shown in the figure below. The spring is unstretched in the equilibrium position.

Hint: Do not forget the gravitational force. "Describe the normal mode motion" means calculate the eigenvalues of the system and sketch the motion of the pendula.

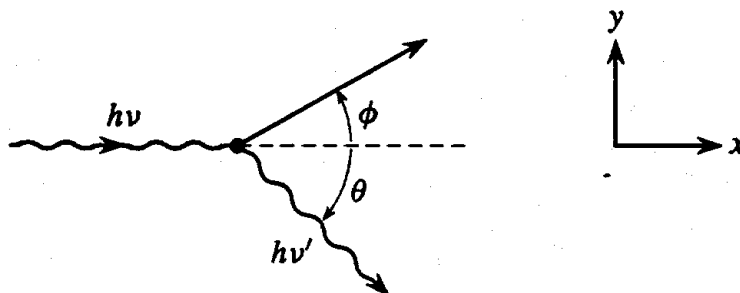


5. The energy of a light quantum (photon) is given by  $E = h\nu$ , where  $h$  is Planck's constant and  $\nu$  is the frequency of the photon; the momentum of the photon is  $h\nu/c$ . Show that if the photon scatters from a free electron (mass  $m_e$ , initially at rest), the scattered photon has an energy

$$E' = E \left[ 1 + \frac{E}{m_e c^2} (1 - \cos \theta) \right]^{-1},$$

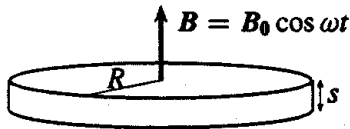
where  $\theta$  is the angle through which the photon scatters. Also show that the electron acquires a kinetic energy

$$T = \frac{E^2}{m_e c^2} \left[ \frac{1 - \cos \theta}{1 + \frac{E}{m_e c^2} (1 - \cos \theta)} \right].$$

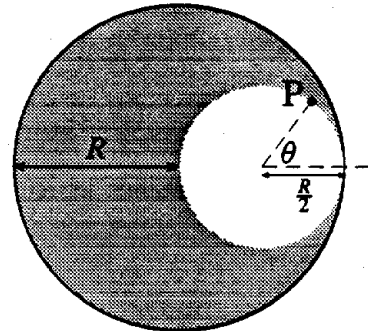


## Electricity and Magnetism

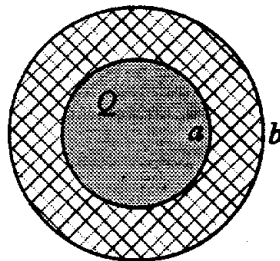
1. One method to determine the conductivity  $\sigma$  of a metal sample is to apply an oscillating magnetic field  $B = B_0 \cos \omega t$  to a disk of the material and measure the power  $\bar{P}$  which goes into heating the disk, averaged over a cycle. Assume you have a disk of thickness  $s$  and radius  $R$ , and the  $B$  field is normal to the cross-sectional area of the disk, as pictured. Determine  $\sigma$  as a function of  $\bar{P}$ ,  $B_0$ ,  $\omega$ ,  $s$ ,  $R$ .



2. A cylindrical conductor of cross-sectional radius  $R$  has a circular hole of radius  $R/2$  cut out of it as shown in the diagram. The current  $I$  is uniformly distributed over the remaining conductor (shaded). Determine the magnitude of the magnetic field at a point  $P$  on the interior edge of the conductor making an angle  $\theta$  with the horizontal.

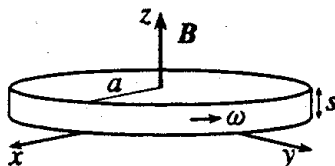


3. A uniform ball of charge  $Q$  and radius  $a$  is surrounded by a thick conducting shell having inner radius  $a$  and outer radius  $b$ , as pictured. The shell has no net charge. Consider the electric potential to be zero at infinity.



Determine the electric potential at the center of the sphere and the electrostatic potential energy of the entire system. Show that the latter result reduces to that of a uniformly charged ball when  $b = a$ .

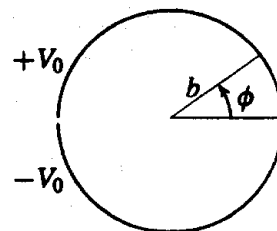
4. A uniform dielectric disk of constant  $\kappa_e$ , radius  $a$ , and thickness  $s$  is rotating with angular velocity  $\omega$  about the  $z$ -axis in a magnetic field  $\mathbf{B} = B_0(s - z)\hat{\mathbf{k}}$ , where the lower cross-sectional area of the disk is in the  $z = 0$  plane, as shown.



Determine the polarization and the volume density of bound charge at a point  $(r, \theta, z)$  within the disk, as well as the surface density of bound charge on all surfaces of the disk. Comment on the relation between the total bound charge inside the disk and that on the surface.

5. An infinitely long, thin-walled circular cylinder of radius  $b$  is split into two half-cylinders. The upper half is fixed at potential  $V = +V_0$  and the lower half at  $V = -V_0$ .

- a) Find the potential inside and outside the cylinder.  
 b) Calculate the charge density as a function of  $\phi$ .  
 (If your answer is in the form of an infinite sum, calculate this sum).



## Quantum Mechanics

1. Consider a 4-state system whose Hamiltonian  $H_0$  has eigenstates labeled  $|\Psi_1\rangle$ ,  $|\Psi_2\rangle$ ,  $|\Psi_3\rangle$ , and  $|\Psi_4\rangle$ , with corresponding energies  $E_1 = 3E_0$ ,  $E_2 = 2E_0$ ,  $E_3 = E_4 = 0$ .

A perturbation  $W$  is applied to the system and is characterized by the following equations:

$$\begin{aligned} W|\Psi_1\rangle &= a|\Psi_2\rangle + b|\Psi_3\rangle + b|\Psi_4\rangle, \\ W|\Psi_2\rangle &= a|\Psi_1\rangle + b|\Psi_3\rangle + b|\Psi_4\rangle, \\ W|\Psi_3\rangle &= b|\Psi_1\rangle + b|\Psi_2\rangle + c|\Psi_3\rangle + d|\Psi_4\rangle, \\ W|\Psi_4\rangle &= b|\Psi_1\rangle + b|\Psi_2\rangle + d|\Psi_3\rangle + c|\Psi_4\rangle, \end{aligned}$$

where  $a$ ,  $b$ ,  $c$ , and  $d$  are positive, real constants with dimensions of energy.

- a) Write down the matrix representation of the full Hamiltonian  $H = H_0 + W$  in the basis defined by the states  $|\Psi_1\rangle$ ,  $|\Psi_2\rangle$ ,  $|\Psi_3\rangle$ , and  $|\Psi_4\rangle$ .  
 b) Find the lowest-order non-zero corrections to the energies of each level caused by the perturbation.

2. A harmonic oscillator is initially in the superposition state:  $|\Psi\rangle = \frac{1}{\sqrt{2}}(|4\rangle + |5\rangle)$ , where  $|n\rangle$  is an eigenstate of the number operator  $N = a^\dagger a$ ,  $N|n\rangle = n|n\rangle$ .

- a) Find  $|\Psi(t)\rangle$  for all  $t > 0$ .  
 b) Determine the expectation values for  $X$  and  $P$ ; verify Ehrenfest's theorem.

3. Consider a particle in one dimension whose Hamiltonian is given by

$$H = \frac{p^2}{2m} + V(x).$$

By calculating  $[[H, x], x]$  prove

$$\sum_{a'} |\langle a|x|a'\rangle|^2 (E_{a'} - E_a) = \frac{\hbar^2}{2m},$$

where  $|a\rangle, |a'\rangle$  are energy eigenkets with eigenvalues  $E_a, E_{a'}$ .

4. The effect on an electron from the passage of a fast particle moving along the  $z$ -axis can be represented by a time-dependent magnetic field of the form

$$\mathbf{B}(t) = B_0 \frac{\tau^2}{t^2 + \tau^2},$$

where  $B_0$  is in the  $z$ -direction and  $\tau = r/v$ , where  $v$  is the velocity and  $r$  the classical distance of closest approach of the charged particle.

Given the Hamiltonian

$$H = \gamma \mathbf{B} \cdot \mathbf{S},$$

( $\gamma$  = gyromagnetic ratio), use the time-dependent Schrödinger equation to evaluate the electron spin state at time  $t$  in terms of the initial state at  $t = -\infty$ .

If the electron is initially polarized along the positive  $x$ -direction, find the final probability that it remains in this state, in terms of  $\omega_0 = \gamma B_0$ .

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + \text{const.}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

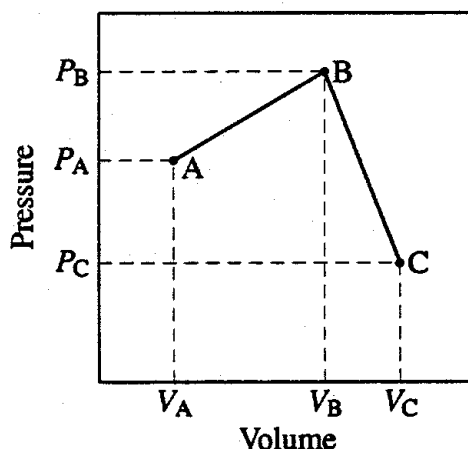
5. A particle of mass  $m$  interacts with a spherically symmetric 3-d potential of the form

$$V(r) = -c \delta(|r| - a),$$

which is a delta function that vanishes unless the particle is precisely a distance  $a$  from the origin. With a proper choice of  $c$ , bound states can exist. Find the wave function and energy for the ground state. (You may stop after obtaining the transcendental equation for the energy; no need to evaluate the normalization constant).

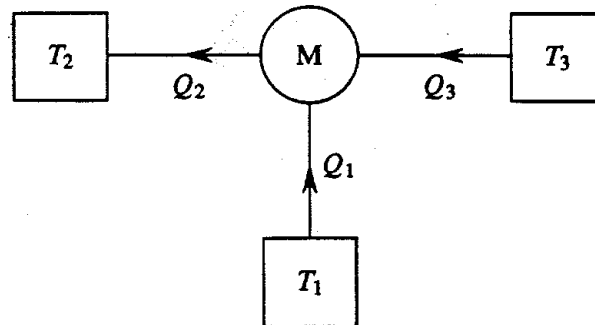
## Thermodynamics

1. **Ideal Gas:**  $N$  molecules of a classical ideal *diatomic* gas (approximated by a symmetric rigid dumbbell) are taken quasistatically from state  $A$  to state  $B$ , and then from state  $B$  to state  $C$  along the straight line paths shown in the diagram of pressure  $P$  versus volume  $V$ .
- What is the work done by the gas in the process  $A \rightarrow B \rightarrow C$ ?
  - What is the change in entropy of the gas in the process  $A \rightarrow B \rightarrow C$ ?
  - Determine the heat absorbed by the gas in the process from  $A \rightarrow B$ .



2. **Equilibrium:** A container with adiabatic walls is separated into two sub-volumes  $A$  and  $B$  by an adiabatic, but movable, partition.  $A$  and  $B$  separately contain  $N$  molecules of the same ideal gas, but are observed to have different temperatures  $T_A$  and  $T_B$  at equilibrium. Now suppose that the partition abruptly loses its adiabatic properties.
- Find the common temperature reached by  $A$  and  $B$  after equilibrium is reestablished.
  - Calculate the total entropy change of the system.
3. **In absorption refrigerators** the energy driving the process is supplied not as work, but as heat from a gas flame at a temperature  $T_3 > T_2$  (the inside of the refrigerator is at  $T_1$ ). Mobile home refrigerators may be of this type, with propane fuel.
- Show that the absorption refrigerator does not violate the second law of thermodynamics for  $T_3 > T_2 > T_1$ .
  - Determine  $Q_1/Q_3$ , the ratio of heat removed to heat supplied, in terms of  $T_1, T_2, T_3$ .
  - Will this refrigerator use less energy than one in which the heat  $Q_3$  is converted to work  $W$  by a Carnot heat engine,  $W$  then being used to drive a conventional Carnot refrigerator? Support your answer.

- c) Will this refrigerator use less energy than one in which the heat  $Q_3$  is converted to work  $W$  by a Carnot heat engine,  $W$  then being used to drive a conventional Carnot refrigerator? Support your answer.



## Statistical Mechanics

- 1. A Frenkel defect in a crystal results from an atom being displaced from a lattice site into an interstitial site (lattice sites are the positions of atoms in the perfect crystal, interstitial sites lie between the lattice sites). Consider a crystal with  $N$  lattice sites and  $M$  interstitial sites. The energy to create a single Frenkel defect is  $\varepsilon$ .**

  - Find the entropy of  $n$  Frenkel defects.
  - Determine  $n$  as a function of temperature for  $n \ll N, M$ .
- 2. Three-orbital Bose system:** Consider a system of  $N$  identical bosons of spin zero, with one orbital at the single-particle energy 0 and two *degenerate* orbitals at the single-particle energy  $\varepsilon$ . Assume  $N \gg 1$ .

  - Find the temperature at which the thermal average population of the lowest orbital is twice the population of a single orbital at  $\varepsilon$ .
  - For very high temperatures, what is the probability that a particle has energy  $\varepsilon$ ?
  - Does this system undergo Bose-Einstein condensation? Briefly explain your reasoning.
- 3. Spin waves:** The Debye theory of the heat capacity of solids treats the crystal as a continuous elastic medium, hence the dispersion relation for phonons (sound wave quanta) is  $\omega = ck$ , where  $c$  is the speed of sound and  $k$  the wave vector magnitude. In a ferromagnetic solid at low temperatures, there exist quantized waves of magnetization called spin waves. The dispersion relation for this type of wave is  $\omega \propto k^2$ .

  - Find the low temperature heat capacity due to spin waves in a ferromagnetic solid.
  - Find the low temperature limit of the entropy due to spin waves.