Physics Comprehensive Exam - Spring 2005

Classical Mechanics

March 22, 2005
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems in Classical Mechanics. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: Code Symbol:

Classical Mechanics problems: 1 2 3 4 5

Rigid Body.

Consider a thin homogeneous rectangular plate of mass M and area ab (with $b > a$) that lies on the (x, y)-plane.

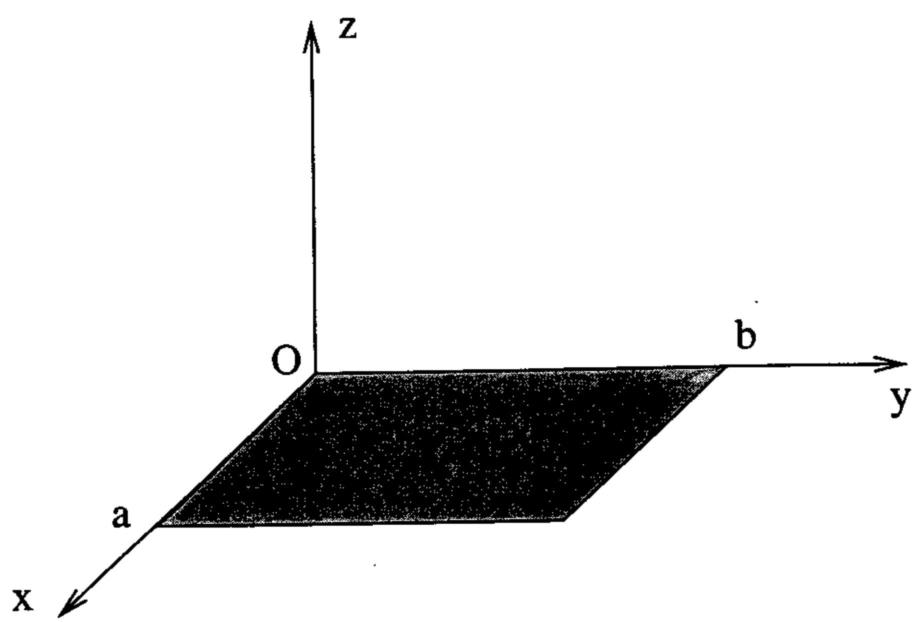
(a) Show that the inertia tensor (calculated in the reference frame with its origin at point O in the figure) takes the form

$$\mathbf{J} = \begin{pmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{pmatrix},$$

and find suitable expressions for A, B, and C in terms of M, a, and b.

(b) Calculate the inertia tensor $\mathbf{I}$ in the center of mass frame by using the Parallel-Axis Theorem.

(c) From the result in Part (b), determine the principal moments of inertia (I_1, I_2, I_3) and the principal axes of inertia (e_1, e_2, e_3).



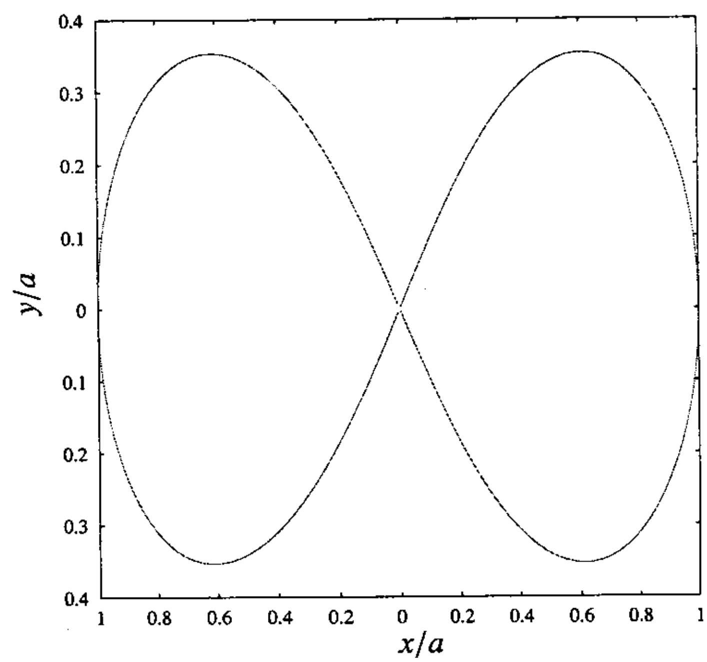
Classical Mechanics 2

Periodic Motion.

A particle of mass m and angular momentum ℓ is observed to undergo periodic motion with its distance $r(\theta)$ from the center of force given by the relation $r^2(\theta) = a^2\cos(2\theta)$. This formula describes a lemniscate of amplitude a, where r and θ represent the usual variables in polar coordinates.

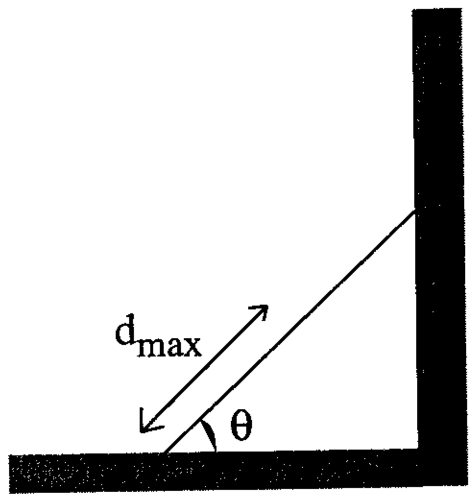
(a) Use the radial acceleration $a_r = \ddot{r} - \dot{\theta}^2 r$ to find the potential energy $U(r)$ that leads to this periodic motion.

(b) Find the period of motion.



Sliding ladder.

A ladder of length L with negligible mass leans against a wall. The ladder forms an angle θ with the floor. The static friction coefficient between the ladder and the floor and between the ladder and the wall is μ. A child of mass m walks up the ladder. What is the maximum distance d_{max} that the child can walk up the ladder before the ladder starts to slip?

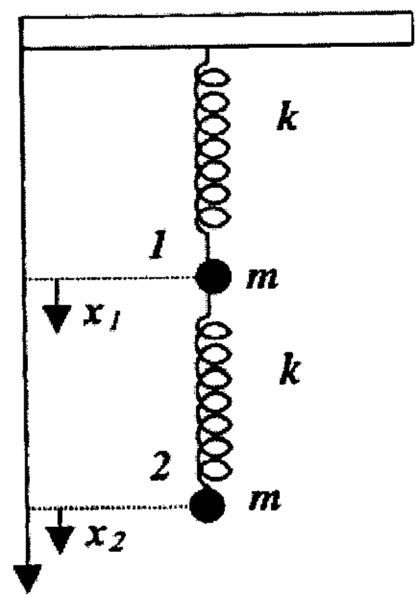


Coupled Oscillators.

Two identical bodies are suspended by two identical springs in series, with spring constants equal to k.

(a) Find the two equations of motion for the two masses m.

(b) Find the general solutions for $x_1(t)$ and $x_2(t)$.



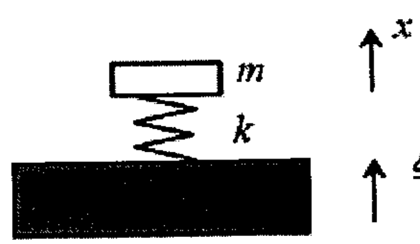
Forced Harmonic Oscillator.

A device of mass m is put on a spring (constant k) bound to a machine that oscillates with a frequency Ω and an amplitude ξ_m ($\xi = \xi_m \cos(\Omega t)$). A damping system acts on the mass m with a force $\mathbf{f} = -\eta\dot{\mathbf{x}}$, where $\mathbf{x}$ is the displacement of m. **Do not neglect gravity!**

(a) Find the relationship between m and x at equilibrium. Write an expression for the force exerted by the spring for a generic m and x.

(b) Find the equation of motion for m. You can use $X = x - x_0$ where x_0 is the equilibrium position. Suggest a complex equation in which the real part is equivalent to the equation of motion.

(c) Find the stationary complex solution with the form $z = z_0 e^{i\Omega t}$. From this solution, find the real amplitude of X as a function of Ω.



Physics Comprehensive Exam - Spring 2005

Electricity & Magnetism

March 23, 2005
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems in Electricity & Magnetism. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: Code Symbol:

Electricity & Magnetism problems: 1 2 3 4 5

Electricity & Magnetism 1

Collision of Charged Particles.

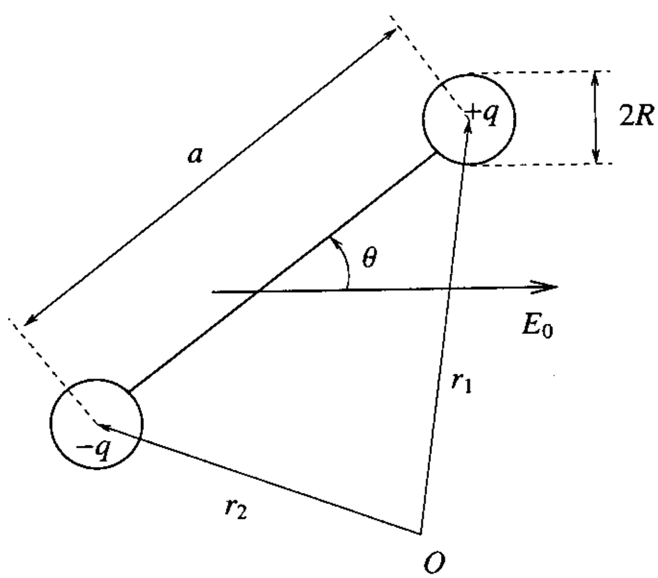
Two colliding non-relativistic particles have charges q_1 and q_2 and masses m_1 and m_2, respectively. Under what condition is the collision of these two particles **not** accompanied by dipole radiation?

Electric pendulum.

Two identical and perfectly conducting spheres of radius R and mass M are connected by a thin and rigid perfectly conducting rod. The system is polarized (as shown in the figure) by placing it in an external electric field $\mathbf{E}_0$ that makes an angle θ with the rod.

(a) Find the dipole moment $\mathbf{p}$ induced in this system. Assume that $R \ll a$, where a is the distance between the centers of the spheres.

(b) Compute the torque $\mathbf{N}$ on the system and the frequency of small oscillations near $\theta = 0$. Neglect the mass and capacitance of the rod. If you could not do part (a), assume that $\mathbf{p} = A(\mathbf{r}_1 - \mathbf{r}_2)\cos\theta$, where A is a constant.



Quasistatic Solenoid.

An infinitely long solenoid has n turns per length wrapped around a cylindrical tube of radius R. The solenoid carries an alternating current $I(t) = I_0 \cos(\omega t)$ with frequency $\omega \ll c/R$. Find the magnetic field outside the solenoid at distances $\rho \ll c/\omega$ from its axis in the lowest non-vanishing order in ω.

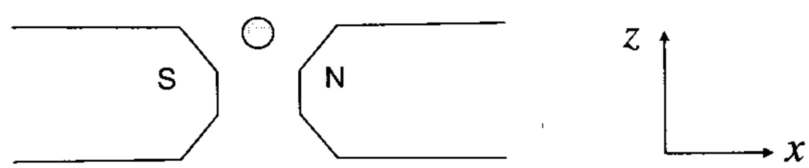
Hint: to avoid a logarithmic divergence, you will need to change the upper limit of a radial integral from ∞ to the wavelength c/ω. This is consistent with the quasistatic approximation.

For reference: in cylindrical coordinates,

$$\nabla \times \mathbf{f} = \frac{1}{\rho}\left[\frac{\partial f_z}{\partial \phi} - \frac{\partial f_\phi}{\partial z}\right]\hat{\rho} + \left[\frac{\partial f_\rho}{\partial z} - \frac{\partial f_z}{\partial \phi}\right]\hat{\phi} + \frac{1}{\rho}\left[\frac{\partial(\rho f_\phi)}{\partial \rho} - \frac{\partial f_\rho}{\partial \phi}\right]\hat{z}.$$

Magnetic Susceptibility.

The relation $\mathbf{M} = \chi_m \mathbf{H}$ defines the magnetic susceptibility of a sample of matter. Faraday discovered that some substances have $\chi_m > 0$ (paramagnetic) and some substance have $\chi_m < 0$ (diamagnetic). The figure below is a side view of a small spherical sample that sits near the pole faces of a permanent magnet. Discuss the relation between the sign of χ_m for this sample and the direction of the magnetic force it feels.

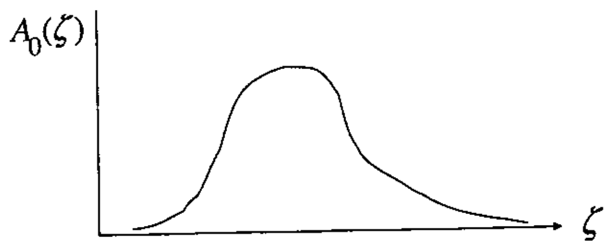


Electromagnetic Pulse.

The vector potential of an electromagnetic pulse is

$$\mathbf{A}(\mathbf{r}, t) = (a\hat{x} + ib\hat{y})A_0(\zeta)\exp(i\zeta),$$

where $\zeta = k(z + ct)$. The envelope function $A_0(\zeta)$ has the form sketched below.



(a) Find the (real) electric field $\mathbf{E}(\mathbf{r}, t)$ and (real) magnetic field $\mathbf{B}(\mathbf{r}, t)$ in the approximation that the envelope function varies slowly over one wavelength of the wave.

(b) Find the linear momentum density carried by the fields of part (a). Explain the physical origin of its algebraic sign.

Physics Comprehensive Exam - Spring 2005

Quantum Mechanics

March 24, 2005
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems in Quantum Mechanics. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: Code Symbol:

Quantum Mechanics problems: 1 2 3 4 5

Quantum Mechanics 1

Stretching Quantum Box.

A particle of mass m is contained in a one-dimensional impenetrable box extending from $x = -L/2$ to $x = L/2$. The particle is in its ground state. The walls of the box are now moved out, symmetrically and instantaneously, to form a box extending from $x = -L$ to $x = L$.

(a) Calculate the probability that, after this sudden expansion, the particle will be in the ground state.

(b) What is the probability that it will be found in the first excited state?

(c) What is the expectation value of the energy after the rapid expansion?

Quantum Mechanics 2

Angular Momentum Measurement.

Let $|\ell, m\rangle$ denote the eigenvectors of L^2 and L_z. A measurement of L^2 and L_z for a free particle yields the values $\ell = 1$ and $m = 1$. Later, a measurement of L_y is made.

(a) What are the possible values of L_y?

(b) Calculate the probability for each of the possible values in part (a).

Quantum Mechanics 3

Three Electron Wavefunction.

Consider the three-electron system of a neutral lithium atom ($Z = 3$). A simple many-body wavefunction can be constructed from one-electron states incorporating the Pauli principle.

(a) Show that the wavefunction for the ground state **cannot** result from the antisymmetrization of the following products:

$$\phi_{1s}^{\alpha}(1)\phi_{1s}^{\alpha}(2)\phi_{1s}^{\alpha}(3)$$

or

$$\phi_{1s}^{\alpha}(1)\phi_{1s}^{\alpha}(2)\phi_{1s}^{\beta}(3),$$

where ϕ_{1s}^{α} corresponds to a normalized $1s$ orbital with spin up and ϕ_{1s}^{β} corresponds to a normalized $1s$ orbital with spin down. The number in parenthesis (1, 2, or 3) is the particle index.

(b) Construct the ground-state wavefunction for neutral Lithium by taking into account possible spin arrangements and proper normalization.

Quantum Mechanics 4

Time-Dependent Perturbation.

Consider a system with Hamiltonian H_0, where

$$H_0\psi_n = E_n\psi_n, \qquad \langle\psi_n|\psi_m\rangle = \delta_{mn}.$$

At time $t = 0$, a perturbation $H'(t)$ is turned on so that the Hamiltonian becomes

$$H = H_0 + H'(t).$$

(a) Let the system be prepared in eigenstate ψ_N. Now suppose that H' is constant (except that it is turned on at $t = 0$ and switched off again at some later time $t = T$). Find the probability that the system will be found in state ψ_M different from ψ_N at time $t > T$.

(b) Consider now a particle of mass m initially in the ground state of the one-dimensional infinite square well. At time $t = 0$, a "brick" is dropped into the well, so that the potential becomes

$$V(x) = \begin{cases} V_0, \text{ if } 0 \le x \le a/2; \\ 0, \text{ if } a/2 \le x \le a; \\ \infty, \text{ otherwise,} \end{cases}$$

where $V_0 \ll E_1$. At time $t = T$, the brick is removed, and the energy of the particle is measured. Find the probability (in first order perturbation theory) that the energy is now E_2.

Possibly useful integral:

$$\int_0^{a/2} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx = \left[\frac{\sin\left(\frac{(m-n)\pi x}{a}\right)}{2(m-n)\pi/a} - \frac{\sin\left(\frac{(m+n)\pi x}{a}\right)}{2(m+n)\pi/a}\right]_0^{a/2}.$$

Quantum Mechanics 5

Spin Hamiltonian.

Consider an electron at rest in the presence of a uniform magnetic field $\mathbf{B} = B_z\hat{\mathbf{z}}$ for which the Hamiltonian is

$$H_0 = \frac{eB_z}{m}S_z.$$

Now turn on a perturbation in the form of a uniform field along the x-direction

$$H' = \frac{eB_x}{m}S_x.$$

(a) Find the matrix elements of H' in the basis of eigenstates of H_0.

(b) Calculate the corrections to the ground state energy to first non-vanishing order in perturbation theory.

(c) Find the **exact** ground state energy of $H_0 + H'$.

(d) Compare your answers from part b and c. Do they agree? If yes, explain why. If not, explain why not!

Physics Comprehensive Exam - Spring 2005

Statistical Mechanics & Thermodynamics

March 25, 2005
9:00 am – 2:00 pm

Instructions:

1. Worh **three** problems in Statistical Mechanics and **one** problem in Thermodynamics. **DO NOT WORK ALL SIX!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: Code Symbol:

Stat. Mech. problems:	1	2	3	4
Thermo problems:	5	6		

Statistical Mechanics 1

Thermal Ionization of Hydrogen Atoms.

Assume that Hydrogen (H) atoms are enclosed in a container of volume V and temperature T. Consider the possibility that H can be thermally ionized into $H^+ + e^-$, under conditions of chemical and thermal equilibrium. Define the ionization energy of H to be E_0.

(a) Find the partition function for each of the three subsystems: H, H^+ and e^-.

(c) Find the fraction of H atoms dissociated at temperature T.

Statistical Mechanics 2

Ultra-Relativistic Gas.

If the particles of a gas have velocities close to the speed of light c, their energy has to be calculated relativistically. In the limit of massless particles (e.g. photons) which travel at the speed of light, this relation between momentum $\mathbf{p}_i$ of particle i and its energy E_i becomes $E_i = cp_i$, where p_i is the magnitude of vector $\mathbf{p}_i$. Thus, the energy of a gas of N of these particles in a box of volume V is

$$H(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N) = c \sum_{i=1}^{N} p_i$$

for all $\mathbf{r}_i$ inside volume V, and $H(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N) = +\infty$, otherwise.

(a) Calculate the partition function of such an ultra-relativistic gas.
(Hint: $\int_0^\infty x^2 \exp(-x) = 2$.)

(b) Calculate the Helmholtz free energy. Use Stirling's formula to express your result in an explicitly extensive form.

(c) Derive the equation of state.

(d) Calculate the internal energy.

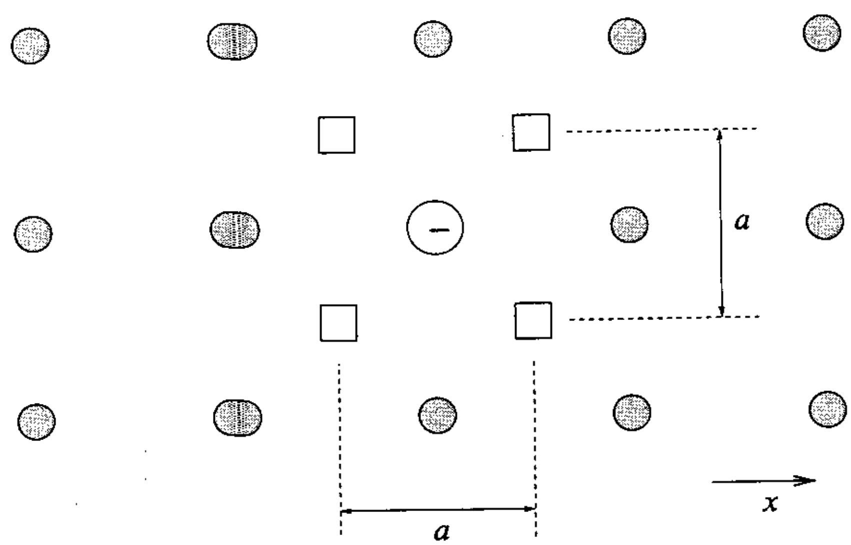
Electric Polarization in Solids with Defects.

A solid made of neutral atoms is at temperature T and contains N negatively charged impurity ions on its surface, which replace regular atoms. For each negative ion with charge $-q_0$ there is one nearby positive ion with charge $+q_0$ that can sit in any of the four distinct positions shown in the figure below as square boxes. When a small electric field $\mathbf{E}$ is applied along the x-direction:

(a) Calculate the partition function;

(b) Calculate the electric polarization $\mathbf{P}$ of the surface;

(c) Discuss your results for $\mathbf{P}$ physically at low and high temperatures.



Statistical Mechanics 4

Lowest Three Molecular Levels.

The three lowest energy levels of a certain molecule are $E_0 = 0$ (the ground state), $E_1 = \varepsilon$ (first excited state), and $E_2 = 10\varepsilon$ (second excited state). Assume a system of N such molecules (with $N \gg 1$), obeying Boltzmann statistics.

(a) Find the average energy $\langle E \rangle$ at temperature T.

(b) Find the contribution of these three levels to the molar specific heat C_V at temperature T.

(c) Find limiting expressions for the C_V obtained above in the high- and low-temperature regimes. Use this information to draw a rough sketch of C_V as a function of T.

Entropy of Ideal Gas and Reservoir.

An ideal gas inside a rigid cylinder of volume V_1 is at pressure P and temperature T_1. The temperature of the system is raised to T_2 by placing it in contact with an infinite reservoir at temperature T_2.

(a) What is the change in entropy of the ideal gas during this process?

(b) What is the change in entropy of the reservoir during this process?

(c) Now assume that the top wall of the cylinder is allowed to act like a piston. Find a process in the T-V plane where the initial and final volumes are both V_1 and the net entropy change of the universe is **zero**. The process may involve adiabatic and/or isothermal volume changes.

Thermodynamics 2

Ideal Gas and Impermeable Membrane.

A thermally insulated cylinder is closed at both ends. The cylinder contains two chambers which are separated by an impermeable membrane that can move without friction. Initially the membrane is clamped in a position where the volume to the left is V_0, while the volume to the right is $3V_0$. Both compartments contain the same monatomic ideal gas. The initial temperature is T_0 on both sides, while the pressure is P_0 to the left and $2P_0$ to the right. The membrane is then released and after some time the system reaches equilibrium.

(a) Find the final temperature of each side.

(b) Find the final volume of each side.

(c) Find the final pressure of each side.