

Physics Comprehensive Exam - Spring 2002

Classical Mechanics 1

Day 1

Classical Mechanics and Thermodynamics

Monday, March 11, 2002
9:00 am – 3:00 pm

Instructions:

1. Do **four** of the five problems in Classical Mechanics and **one** of the two problems in Thermodynamics.
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - **NOT YOUR NAME** - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: _____

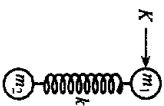
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Classical Mechanics problems: 1 2 3 4 5

Thermodynamics Problems: 1 2

Two point masses, with masses $m_1 = m_2 = m$, are at rest on a frictionless horizontal surface. The masses are connected by a spring of unstretched length ℓ_0 and force constant k . A momentum impulse $K\hat{x}$ is given to m_1 at time $t = 0$, in a direction perpendicular to the spring. Assume that the spring always lines up along the connecting length, i.e. there is no bending.

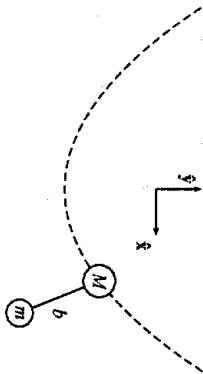
- (a) For $t = 0^+$ (immediately after the impulse), determine the following:
 - i. The center-of-mass velocity V .
 - ii. The angular momentum about the center of mass, L .
 - iii. The velocity v_1 of mass m_1 .
 - iv. The velocity v_2 of mass m_2 .
- (b) Find equations for E and L in terms of the coordinates (r, θ) and $(\dot{r}, \dot{\theta})$ of m_1 relative to the center of mass.
- (c) Determine the quartic equation satisfied by the maximum separation length ℓ_{max} between the two mass points during the motion that follows the impulse.



Classical Mechanics 2

A particle, mass M , moves along the curve $y = \frac{1}{2}cx^2$, where c is a positive constant and the gravitational acceleration is $-g\hat{y}$. Another particle, mass m , is suspended from the first particle by a light rod of length b , which is free to rotate in the x - y plane. Assuming that the second particle only moves in the x - y plane:

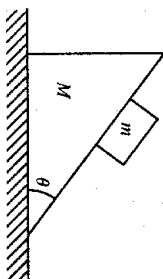
- Set up the Lagrangian for this system.
- Obtain the equations of motion.



Classical Mechanics 3

A block of mass m is placed on top of a heavier block of mass M , on a surface that makes an angle θ with the horizontal. Mass M slides without friction on a horizontal floor. The coefficient of static friction between the two blocks is μ_s while the coefficient of kinetic friction is μ_k .

- Determine the critical angle θ_c beyond which mass m will begin to slide.
- For $\theta > \theta_c$, find the x -component of acceleration for mass m .
- What is the minimum coefficient of friction μ_{k1} between mass M and the floor that would prevent the motion of mass M ?



Classical Mechanics 4

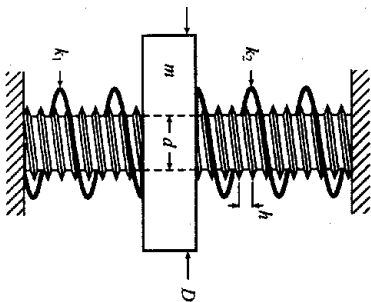
A deuterium nucleus (neutron + proton) with kinetic energy $K_d = 3.25 \text{ MeV}$ collides with an identical stationary nucleus. The collision results in a nuclear reaction:



where the ${}^3\text{He}$ nucleus contains 2 protons and 1 neutron. In the reaction, an energy W is converted into additional rest mass of the products. Calculate W , if the neutron is motionless from the observer's point of view.

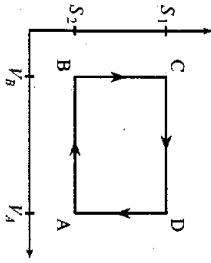
Classical Mechanics 5

A cylindrical nut with internal diameter d and external diameter D can be screwed without friction on a threaded rod. The thread pitch is h , the mass of the nut is m . Motion of the nut is limited by two springs, of stiffnesses k_1 and k_2 . The ends of the springs push on the upper and lower faces of the nut, but they are not fixed to it; the contacts are frictionless. Assuming that the depth of the threads is negligible, find the period of oscillation of the nut.



Thermodynamics 1

The operation of a gasoline engine is roughly similar to the Otto cycle, shown here on an entropy-volume diagram:



- AB Adiabatic compression.
- BC Isochoric heating (combustion of gasoline).
- CD Adiabatic expansion (power stroke).
- DA Isochoric cooling.

Calculate the efficiency of the cycle for an ideal gas as a function of the compression ratio V_A/V_B and the heat capacity per particle, $c_{v,m}$.

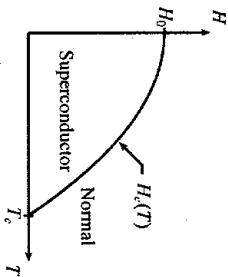
Thermodynamics 2

A type I superconductor in its superconducting phase is a perfect diamagnet, i.e. all magnetic flux is excluded from inside the superconductor. Consider a type I superconductor with a parabolic coexistence curve separating the (uniform) superconducting and normal phases. H is the external magnetic field and T is the temperature. Ignore the magnetization of the normal phase. The critical line is given by:

$$H_c = H_0 + aT + bT^2$$

- (a) Why must a be zero?
- (b) In terms of H_0 and T_c , calculate the latent heat per unit volume as a function of T along the critical line.
- (c) Along the critical line, calculate the discontinuity in the specific heat per unit volume at constant H .

Note: You may find it helpful to recall that $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ in SI units or $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ in CGS units, where \mathbf{M} is the magnetization.



Physics Comprehensive Exam - Spring 2002

Electricity & Magnetism I

Day 2

Electricity & Magnetism and Statistical Mechanics

Tuesday, March 12, 2002

9:00 am - 3:00 pm

Instructions:

1. Do **four** of the five problems in Electricity & Magnetism and **one** of the two problems in Statistical Mechanics.
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - **NOT YOUR NAME** - at the top of each sheet of paper you hand in.
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GOOD LUCK!

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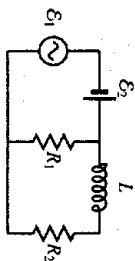
Code Symbol: _____

Electricity & Magnetism problems: 1 2 3 4 5

Statistical Mechanics problems: 1 2

An AC source $\mathcal{E}_1 = (20 \text{ V}) \cos 2\pi ft$ is in series with a battery $\mathcal{E}_2 = 16 \text{ V}$ connected to a circuit consisting of resistors $R_1 = 10 \Omega$ and $R_2 = 8 \Omega$, and an inductor $L = 6 \text{ mH}$. Find the average power dissipated in R_1 and R_2 for

- (a) $f = 100 \text{ Hz}$.
- (b) $f = 800 \text{ Hz}$.



Electricity & Magnetism 2

Quantum Capacitance. Let d be the separation between two infinite, parallel, perfectly conducting plates. The lower surface of the upper plate has charge/area $\sigma_1 > 0$. The upper surface of the lower plate has charge/area $\sigma_2 > 0$. At a distance L above the lower plate, there is an infinite sheet of charge with charge/area $\sigma_0 = -(\sigma_1 + \sigma_2)$. If we approximate the latter by a two-dimensional electron gas, statistical mechanics tells us that the energy/area of the sheet is $\epsilon_0 = \pi^2 \hbar^2 \sigma_0^2 / 2m e^2$.

For constant σ_1 , show that the total energy/area of the system is minimized when

$$\sigma_2 = -\sigma_1 \frac{C_2}{C_1 + C_0}$$

where C_2 is a characteristic geometric capacitance and C_0 is the "quantum capacitance" of the 2D electron gas. Discuss the classical limit of C_0 and confirm that the corresponding value for σ_2 makes sense.

Electricity & Magnetism 3

Wire radiation. A very long (infinite) wire with a very small cross-sectional area A is coincident with the z -axis. The wire carries a current $I(t) = I_0 e^{-i\omega t}$.

- Use the Poynting vector to determine the dominant dependence of the radiation fields on the variable ρ (cylindrical coordinates) that measures the distance perpendicular to the wire.
- The radiation magnetic field from a time-dependent point electric dipole $\mathbf{p}(t)$ at the origin is

$$\mathbf{B}_{\text{rad}}(\mathbf{r}, t) = -\frac{\mu_0 \dot{\mathbf{p}}}{4\pi r} \times \frac{d^2 \mathbf{p}(t - r/c)}{dt^2}$$

Explain why the radiation magnetic field from the wire can be regarded as a superposition of fields like the one above.

- Carry out the superposition and confirm the answer you got in part (a). Simplify the integral you get by assuming that $z \ll \rho$ (even if it appears not to be justified) and then justify the approximation afterwards.

Useful fact: $\int_{-\infty}^{\infty} dz e^{-\alpha z^2} = \sqrt{\pi/\alpha}$

Electricity & Magnetism 4

A conductor is charged through repeated contact with a metal plate. Before each contact, the metal plate is recharged to total charge Q . If q is the charge on the conductor after the initial contact and removal, what is the maximum charge on the conductor after many contacts?

Electricity & Magnetism 5

A small circular wire loop having radius r_1 lies at the origin and is oriented so that its normal lies along the z -axis. A second small circular wire loop of radius r_2 also has its normal parallel to the z -axis and lies in the x - z plane, at position $(\ell, 0, \ell)$. Find the mutual inductance between the two wire loops. You may assume that ℓ is much greater than the radius of either loop.

Note: The magnetic field \mathbf{B} due to a dipole \mathbf{m} at the origin is

$$\mathbf{B} = \frac{\mu_0}{4\pi r^3} (3(\mathbf{m} \cdot \hat{r})\hat{r} - \mathbf{m})$$

Statistical Mechanics 1

In a uniform magnetic field \mathbf{B} taken along the z -direction, the projection of the orbital motion of an electron on the xy -plane makes a circular motion with angular frequency $\omega_0 = eB/m$ (SI units) on account of the Lorentz force $e\mathbf{v} \times \mathbf{B}$. This circular motion can be regarded as a quantized harmonic oscillation. Hence the energy of the electron is given by the following formula:

$$E(\ell, p_z) = \hbar \frac{eB}{m} \left(\ell + \frac{1}{2} \right) + \frac{p_z^2}{2m} \quad (\ell = 0, 1, 2, \dots)$$

where p_z is the momentum in the z -direction (along the field).

Consider a system of N such electrons in a magnet field \mathbf{B} , at a temperature high enough that Boltzmann statistics hold. Show that the magnetic moment of the system is given by

$$M = -N\mu_B \left(\coth x - \frac{1}{x} \right),$$

where $\mu_B = e\hbar/2m$, is the Bohr magneton and $x = \mu_B B/kT$. The spin of the electron and the Coulomb interactions between electrons are neglected.

(Hint: Note that the energy level specified by ℓ has a high degeneracy which originates from a coalescence of the zero-field levels. The energy levels in zero magnetic field that lie between the limits

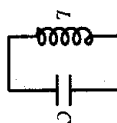
$$2\mu_B B\ell < (p_z^2 + p_z^2)/2m < 2\mu_B B(\ell + 1)$$

coalesce into the level $2\mu_B B(\ell + 1/2)$ in the magnetic field B . Also recall that the magnetic moment of a current loop is current \times area.)

Statistical Mechanics 2

The circuit shown consists of inductance L and capacitance C . What is the root-mean-square noise voltage across the capacitor at temperature T ,

- when T is very large?
- when T is very small?



Physics Comprehensive Exam - Spring 2002

Quantum Mechanics I

Day 3

Quantum Mechanics
and
Statistical Mechanics/Thermodynamics

Wednesday, March 13, 2002
9:00 am - 3:00 pm

Instructions:

1. Do **four** of the five problems in Quantum Mechanics and **one** of the two problems in Statistical Mechanics/Thermodynamics.
2. On this cover sheet, circle the problems that you have chosen to hand in.
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GOOD LUCK!

Name: _____

Code Symbol: _____

Quantum Mechanics problems: 1 2 3 4 5

Statistical Mechanics problem: 3

Thermodynamics problem: 3

Matrix Elements. Consider the Hamiltonian $H = \frac{P^2}{2m} + V(r)$. Show that

$$\langle k|p|l \rangle = im \frac{E_k - E_l}{\hbar} \langle k|V(r)|l \rangle$$

and

$$\langle k|p|l \rangle = \frac{\hbar}{E_k - E_l} \langle k|V(r)|l \rangle$$

where $|k\rangle, |l\rangle$ are the eigenstates of H . Interpret the second result physically.

Quantum Mechanics 2

A beam of silver atoms is produced by evaporation in a furnace at $2,227^\circ\text{C}$. The beam is collimated and travels in the $+x$ direction through the inhomogeneous field of a Stern-Gerlach Magnet. The gradient of the magnetic field in the $+z$ direction is 140 T/m . The length of the magnet is 15 cm , and a screen is placed a distance of 1 m from the back end of the magnet.

Calculate the separation of the spin up and spin down components of the beam on the screen.

You must clearly state any assumptions that you make.

Atomic mass of silver: 107.87

Atomic mass unit: $1.661 \times 10^{-27}\text{ kg}$

Bohr magneton: $e\hbar/2m = 9.274 \times 10^{-24}\text{ J/T}$

Quantum Mechanics 3

A free electron is put inside a cavity in which there exist two magnetic fields: A constant homogeneous field \mathcal{H}_0 in the z direction, and a field \mathcal{H}' rotating in the x - y plane. By components:

$$\mathcal{H}'_x = 0;$$

$$\mathcal{H}'_y = 0;$$

$$\mathcal{H}'_z = \mathcal{H}_0;$$

$$\mathcal{H}'_x = \mathcal{H}' \cos \omega t;$$

$$\mathcal{H}'_y = \mathcal{H}' \sin \omega t;$$

$$\mathcal{H}'_z = 0;$$

At time $t = 0$, the electron spin is in the $+z$ direction, and the field \mathcal{H}' is switched on. Determine $P(t)$, the probability as a function of time for the electron spin to be inverted to the $-z$ direction.

Spin matrices:

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Quantum Mechanics 4

Anharmonic Oscillator. Consider a quantum mechanical oscillator defined by the following potential:

$$V(x) = \frac{1}{2}m\omega^2x^2 + \epsilon_1\left(\frac{x}{l}\right)^3 + \epsilon_2\left(\frac{x}{l}\right)^4; \quad l = \sqrt{\frac{\hbar}{m\omega}}$$

- (a) With $\epsilon_1 = 0$, qualitatively compare the energy eigenstates for ϵ_2 positive and negative.
 (b) Find the correction to the energy of the n -th eigenstate to lowest nonzero order in $\epsilon_1, \epsilon_2 > 0$. Recall that for $\epsilon_1 = \epsilon_2 = 0$ the energies are $E_n = (n + 1/2)\hbar\omega$.

Quantum Mechanics 5

Angular Momentum. A particle moves in two dimensions in a circularly-symmetric potential. The wavefunction for the particle is

$$\psi(r, \theta) = Ne^{-\lambda r}(1 + e^{2i\theta})$$

where λ is a constant and N is the normalization.

- (a) What is the probability that a measurement of the angular momentum L_z would yield a value 0?
 (b) What is the mean value of L_z ?
 (c) If $\vec{L} = \vec{r} \times M\vec{v}$, where $\vec{v} = (v_r, v_\theta)$ in cylindrical coordinates, find the mean orbital speed $\langle |v_\theta| \rangle$.

Statistical Mechanics 3

The normal modes of a violin string with a fundamental angular frequency ω_0 may be considered as a set of noninteracting harmonic oscillators. Treat each mode as an independent quantum harmonic oscillator (drop the zero point energy for each oscillator).

- Express the average vibration energy of the system at temperature T as an infinite series. Remember that there are two normal modes for each allowed wavelength.
- In the limit $kT \ll h\omega_0$, express the thermal energy of the system in closed form as a function of T .

Thermodynamics 3

Find the evaporation time of a spherical water droplet of radius $a = 1$ mm in air with relative humidity $f = 40\%$ at $T = 20^\circ\text{C}$. At this temperature the mass density of the saturated vapor is $\rho_{\text{sat}} = 1.7 \times 10^{-5}$ g cm $^{-3}$, and the vapor diffusion coefficient is $D = 0.22$ cm 2 s $^{-1}$.