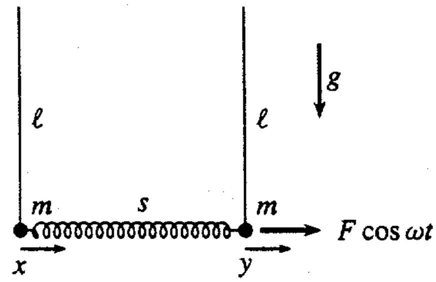
Comprehensive Exam — Spring 2001

Classical Mechanics

1. One pendulum of a pair of coupled pendula is driven by the horizontal force $F\cos\omega t$. Each mass m experiences the same frictional damping rate r (force per unit velocity), and a linear restoring force due to the spring, with stiffness s. The pendula rods have negligible mass, and their lengths, ℓ, are assumed sufficiently long that $\ell \gg \max(x, y)$, where x and y are the horizontal displacements of the masses.

   Find the normal mode frequencies Ω_1 and Ω_2 of oscillation. In the steady state, discuss the relative motion of the masses in the limit $r \to 0$, as a function of ω.



2. A projectile is launched from the surface of the Earth at an angle α with respect to the local vertical direction, i.e., the Earth's radial direction. The initial velocity is $v_0 = \sqrt{gR}$, where R is the radius of the Earth.

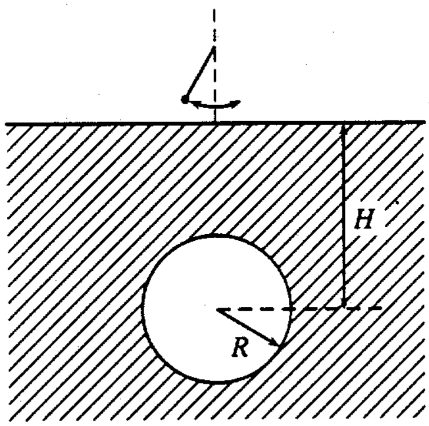
   Find the maximum height (distance from the surface of the Earth) it can reach. Neglect air resistance and the rotation of the Earth.

3. A mathematical pendulum on the Earth's surface is positioned above a spherical underground cavity. The cavity's diameter is $2R = 10\,\text{m}$, and it is $H = 15\,\text{m}$ under ground (see fig).

   (a) Estimate the change in the period of the pendulum, $\Delta T/T$, due to the cavity's presence. Take the density of the soil to be $\rho = 2\,\text{g/cm}^3$.

   (b) Estimate $\Delta T/T$ if the cavity is replaced by an underground tunnel with the same R and H.

   $G_N = 6.67 \cdot 10^{-11}\,\text{m}^3\text{kg}^{-1}\text{s}^{-2}$



4. **Relativistic Rocket** Obtain the equation of motion for a rocket in which either the rocket or the exhaust gases or both may be travelling with relativistic velocities. After obtaining the equation of motion solve it for the special cases where the rocket is accelerated from rest with *no external* force acting on it. Give a physical interpretation of the results in the non-relativistic and extreme relativistic cases!

5. Two bodies subject only to their mutual gravitational attraction orbit in circles about their common center of mass with period T. Suddenly they are stopped in their tracks. Then they are released and allowed to fall towards each other. How long after they are released do they collide?

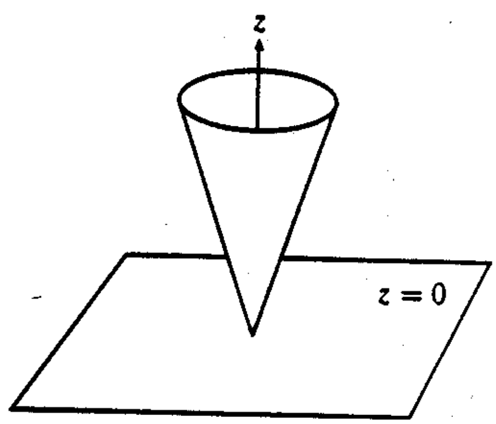
Electricity and Magnetism

1. **TEM Spherical Waves** Consider time-harmonic $[\exp(-i\omega t)]$ solutions to the Maxwell equations in vacuum where the fields are *independent* of the azimuthal variable ϕ. TEM solutions of this type also satisfy $E_r = B_r = 0$.

   (a) Show that these constraints decouple the Maxwell curl equations into two subsets, each of which describes a different type of TEM wave.

   (b) Begin with the Maxwell divergence equations and find the general solution for $\boldsymbol{E}(r, \theta, t)$ and $\boldsymbol{B}(r, \theta, t)$ for each of the two TEM wave types.

   (c) Let the apex of an infinite, solid conducting cone touch the conducting half-space $z < 0$ as shown at right. Explain why this structure can be used to guide one of the TEM waves found in part (b) but not the other.



Useful information:

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi},$$

$$\nabla \times \boldsymbol{A} = \hat{\boldsymbol{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{\boldsymbol{\theta}}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]$$
$$+ \hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right].$$

2. **Electric Field of a Pulsar** A popular model for a pulsar is a highly conducting sphere of radius R that rotates with constant angular velocity Ω around the z-axis. For $r < R$, a "frozen-in" magnetic dipole field (coming from a point dipole at the origin, with magnetic moment $\boldsymbol{m} = m\hat{z}$) rotates along with the sphere. The conductivity arises from a collection of free electrons and positive ions (confined to $r < R$) that also rotates along with the sphere.

   (a) Explain why the pulsar must have an internal electric field.

   (b) Assume the magnetic field arises from a point dipole at the origin. Show that

   $$\boldsymbol{E} = \frac{\mu_0 m\Omega}{4\pi r^2}\left(\sin^2\theta\,\hat{\boldsymbol{r}} - 2\sin\theta\cos\theta\,\hat{\boldsymbol{\theta}}\right) \quad \text{for } r < R.$$

   (c) Find the total charge of the pulsar.

3. Given an interface between two non-absorbing dielectrics with different refractive indices. Using time-reversal symmetry, show that the coefficient of reflectivity, the ratio of the reflected electric field to the incident electric field, for light incident from one side of the boundary has opposite sign from the coefficient when light is incident from the other side of the interface.
   Which coefficient of reflectivity (reflection off a material of higher or off of lower refractive index) carries with it a negative sign, indicating a 180° phase reversal of a wave upon reflection.

4. Starting with the inhomogeneous scalar wave equation

   $$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \kappa\frac{\partial^2 P}{\partial t^2},$$

   where E is the electric field, P is the polarization, and κ is a constant, make the slowly varying envelope approximation for a nonlinear-optical process that generates light at frequency 2ω from light at ω:

   Assume $E(x,t) = \operatorname{Re} E_0(x)\exp i[(2\omega/c)n(2\omega)x - 2\omega t]$, where c is the speed of light in vacuum, $E_0(x)$ is independent of time and varies slowly on a length scale of the wave length of light. Assume that $P(x,t) = \operatorname{Re} P_0\left\{\exp i[(\omega/c)n(\omega)x - \omega t]\right\}^2$, where P_0 is c constant. $n(\omega)$ and $n(2\omega)$ are the refractive indices of light at the respective frequencies. Make reasonable approximations, and simplify the wave equation to a simple first-order differential equation.

   If $E_0(x) = 0$ at $x = 0$, show that $E(x,t)$'s irradiance at $x = L$ will be proportional to $L^2\mathrm{sinc}^2[(\delta k)L/2]$, where $\mathrm{sinc}(x) = \sin(x)/x$, and $\delta k = (2\omega/c)[n(2\omega) - n(\omega)]$.

5. It may have occurred to you that since parallel currents attract, the current within a single wire should contract into a tiny concentrated stream along the axis (in a plasma, this is known as the "pinch effect"). Yet in practice the current typically distributes itself quite

5. It may have occurred to you that since parallel currents attract, the current within a single wire should contract into a tiny concentrated stream along the axis (in a plasma, this is known as the "pinch effect"). Yet in practice the current typically distributes itself quite uniformly over the wire. How do you explain this?
What integral formula relating the charge and current densities within the wire does you explanation entail? If the current density is uniform, can the wire be electrically neutral?

Quantum Mechanics

1. Consider the Hamiltonian $H = H_0 + V$, where

$$H_0 = \tfrac{1}{2}\hbar\omega(a^\dagger a + aa^\dagger), \qquad V = \lambda(aa + a^\dagger a^\dagger),$$

and λ is real. The operators a and $a^\dagger$ obey the usual harmonic oscillator commutation relations $[a, a^\dagger] = 1$.

   (a) Treat V as a perturbation on H_0.
      i. Find the first order correction to all the energy eigenvalues.
      ii. Compute the ground state energy to second order in λ.
      iii. Find the lowest energy eigenfunction to first order in λ.

   (b) Solve H exactly:
      i. Introduce operators b and $b^\dagger$ defined by

$$a = u\,b + v\,b^\dagger, \quad a^\dagger = u\,b^\dagger + v\,b,$$

      where u and v are real numbers. Show that if $u^2 - v^2 = 1$, then $[b, b^\dagger] = 1$.
      ii. Rewrite H in terms of b and $b^\dagger$. Show that a judicious choice of u and v allows you to diagonalize the Hamiltonian.
      iii. Find the eigenvalues of H.
      iv. Compare the lowest energy eigenvalue with what you found in part (a).

2. In NMR (nuclear magnetic resonance) a technique called *pulse refocusing* is often used, which achieves that

$$u_x^\dagger e^{-ia\sigma_z t}\, u_x = e^{ia\sigma_z t},$$

with a being a constant, σ_x, σ_z the usual Pauli matrices, and $u_x = \exp(-i\pi\sigma_x/2)$.
Prove this identity.

3. For deuterium (^{2}H) and tritium (^{3}H), compute numerically (in Å) the wave length of the H_α-line of the Balmer series, given the wave length $\lambda_\alpha = 6562.79$ Å for hydrogen (^{1}H). $m_p/m_e \approx m_n/m_e \approx 1838$

4. Let $x \in [-L, L]$ and $H = \dfrac{p^2}{2m} - g\delta(x)$, with $\Phi(\pm L) = 0$.

   Prove that there is one bound state given by

$$\Phi_\kappa = \sqrt{\frac{\kappa}{1 - 4\kappa L e^{-2\kappa L} - e^{-4\kappa L}}}\,\left(e^{-\kappa|x|} - e^{-2\kappa L} e^{\kappa|x|}\right),$$

   where $g = \frac{\hbar^2}{m}\kappa \coth(\kappa L)$ and $E_\kappa = -\frac{\hbar^2\kappa^2}{2m}$.

   Prove that the odd-parity states are

$$\Phi_n(x) = \frac{1}{\sqrt{L}} \sin k_n x, \quad \text{where } k_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots, \text{ and } E_n = \frac{\hbar^2 k_n^2}{2m}.$$

   Why do these states show no g- dependence?

5. Two identical non-interacting particles are moving in a three-dimensional isotropic harmonic oscillator. Find the degeneracies of the three lowest energy levels for

   (a) particles of spin 1/2.

   (b) particles of spin 1.

Thermodynamics

1. The Euler formula for the thermodynamical potential, Ω, is

$$PV = -\Omega \tag{1}$$

   [The grand canonical partition function, Z_G, is related to Ω by $Z_G = \exp(-\beta\Omega)$, $\beta = 1/k_B T$]; $\Omega = \Omega(T, V, \mu)$ and

$$P = -\left(\frac{\partial\Omega}{\partial V}\right)_{T,\mu}. \tag{2}$$

   Plugging (1) into (2) yields

$$P = P + V\left(\frac{\partial P}{\partial V}\right)_{T,\mu}.$$

   Explain how this can be true and give an independent argument for it.

2. Helium-3 (^{3}He) at low pressures can remain in the liquid phase down to absolute zero ($T = 0$ K). The lowest solidification pressure is 28.9 atm, ($T \neq 0$!).
The molar entropy in the liquid phase at low temperatures can be expressed as $S = RT/T_0$, ($T_0 =$ 0.22 K).
The molar entropy of the solid phase is temperature-independent and equal to $S = R\ln 2$.
The molar volume difference in the liquid and the solid phase is $\Delta V = V_{\text{liquid}} - V_{\text{solid}} \approx 1.25\,\text{cm}^3/\text{mol}$.

Using this information, calculate

(a) the temperature $T_{\min}$, corresponding to the minimum in melting pressure, $p_{\min} = 28.9$ atm;

(b) the temperature dependence of the melting latent heat (sketch it);

(c) the solidification pressure at $T = 0$.

3. **2D Hall Conductance** The differential free energy per unit area can be written $dF = -B\,dM - S\,dT + \mu\,dN$, where M is the magnetic moment per unit area and N is the number of electrons per unit area. An important observable for this system is the so-called *Hall conductance*

$$g = \frac{e}{c}\left(\frac{\partial N}{\partial B}\right)_{T,\mu}.$$

Show that an equivalent formula for this quantity is

$$g = -\frac{e}{c}\left(\frac{\partial M}{\partial \mu}\right)_{T,B}.$$

Statistical Mechanics

1. For capillary waves on the surface of a liquid of molecular weight M, density ρ, and surface tension σ, the relation between the frequency ν and the wave length λ is

$$\nu^2 = \frac{2\pi\sigma}{\rho\lambda^3}.$$

Use an approach analogous to the Debye theory of heat capacities in solids to find the following:

(a) An expression for the surface energy E as a function of temperature for low temperatures.

(b) An expression for the analog to the Debye temperature, i.e., the temperature below which the system must be in order for the expression in part (a) to be valid. For this calculation, you may assume that each atom at the surface of the liquid possesses one degree of freedom.

You may find the following integral useful:

$$\int_0^\infty \frac{x^{\frac{4}{3}}}{\exp(x) - 1}\, \mathrm{d}x \approx 1.68.$$

2. An ideal gas of atoms with mass m fills a container with volume V, which has thin, thermally insulated walls. In one of the container's walls there is a hole with area S through which atoms escape from the container into the surrounding vacuum. The size of the hole is much smaller than the typical size of the container and the atoms' mean free path.
Find the dependence of the gas temperature in the container, $T(t)$, if the initial temperature if T_0. Neglect the heat capacity of the container walls.
(Hint: The number flux through the hole is $J = \frac{1}{4}\, S\, n\, \langle v \rangle$).

3. Electrons in thin metal films can often be treated as a two-dimensional ideal gas. Using Maxwell's distribution, find the ratio of the most probable velocity, v_{mp}, to the mean-square velocity, $\sqrt{\langle v^2 \rangle}$.
v_{mp} is the velocity when the distribution function is at maximum.