Physics Comprehensive Exam - Fall 2004

Classical Mechanics

August 10, 2004
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems in Classical Mechanics. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: Code Symbol:

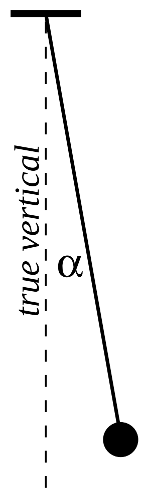
Classical Mechanics problems: 1 2 3 4 5

Plumb Line.

In construction work, a practical means of establishing a vertical reference line is the use of a "plumb line"—a mass hanging (in equilibrium) from a long vertical cord. However, due to the centrifugal effects of the rotating Earth, a plumb line will not hang in a truly vertical direction.

(a) Find an expression for the angular deviation α from vertical, for a plumb line at latitude λ $[\, 0 \le \lambda \le \frac{\pi}{2} \,]$ in the northern hemisphere. Be sure to indicate the direction of the deviation (N, S, E, or W) from the vertical.

(b) Determine the approximate latitude at which the *maximum* deviation would occur, and estimate the magnitude of the deviation at that latitude.



Classical Mechanics 2

Elastic Central Force.

A particle of mass m moves in three dimensions under the influence of a central force which is "spring-like": $F_r = -k\,r$.

(a) Construct a Lagrangian for this system, and determine the equations of motion for the system. What are the obvious constants of the motion?

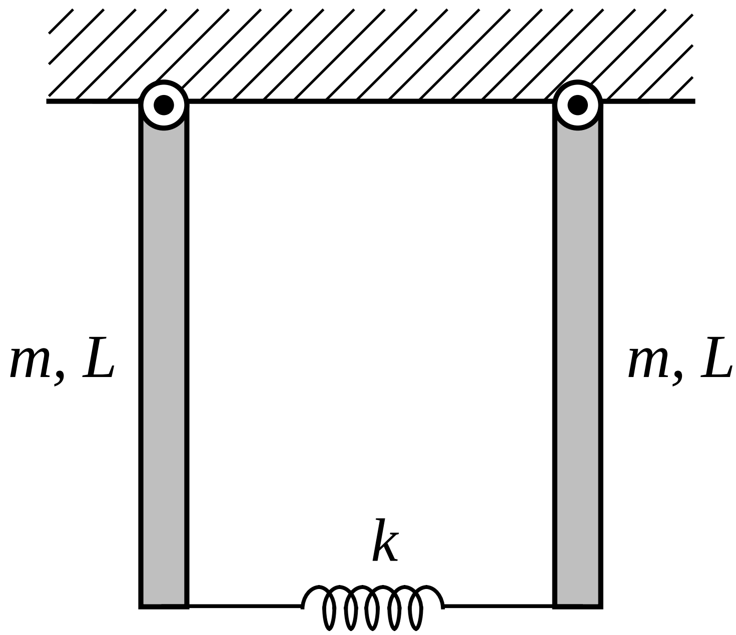
(b) Suppose the particle is in a circular orbit. Find an expression for the radius of the orbit, in terms of the relevant constant(s) of the motion.

(c) Prove that the circular orbit in (b) is stable under small radial deviations: $r = r_0 + \delta$. Determine the frequency Ω of small radial oscillations.

(d) Compare your answer in (c) to the (approximately constant) angular speed of the particle in its (approximately circular) orbit, and comment on whether or not the orbit is closed.

Coupled Pendula.

Two identical rods of mass m and length L hang from pivots, and are connected together at the bottom by a spring with elastic constant k. The equilibrium length of the spring is such that at rest, the two rods hang straight down. Find the frequencies of the two normal modes of oscillation for his system. For each mode, sketch the behavior of the oscillating rods.



Classical Mechanics 4

Space Travel.

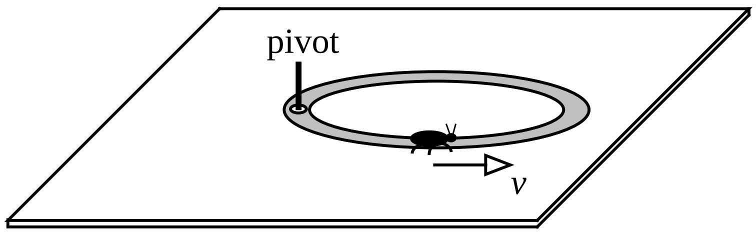
Derive the “rocket equation” describing the motion of a rocket that accelerates by ejecting fuel with velocity u relative to the rocket. Using this equation, find the velocity of the rocket at which its kinetic energy is maximal, assuming the rocket starts at rest and moves in free space.

Insect Walkabout.

A thin ring of mass M and radius R lies flat on its side on a frictionless table. The ring is allowed to rotate freely about a pivot passing through a fixed point on the ring itself. A bug of mass m walks around the ring with speed v, starting at the pivot. Assuming the ring is initially at rest, what is the rotational velocity when the bug is:

(a) halfway around the ring?

(b) back to the pivot point?



Physics Comprehensive Exam - Fall 2004

Electricity & Magnetism

August 11, 2004
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems in Electricity & Magnetism. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: Code Symbol:

Electricity & Magnetism problems: 1 2 3 4 5

Plasma & Plasma Sheath.

A plasma is described by the dielectric function

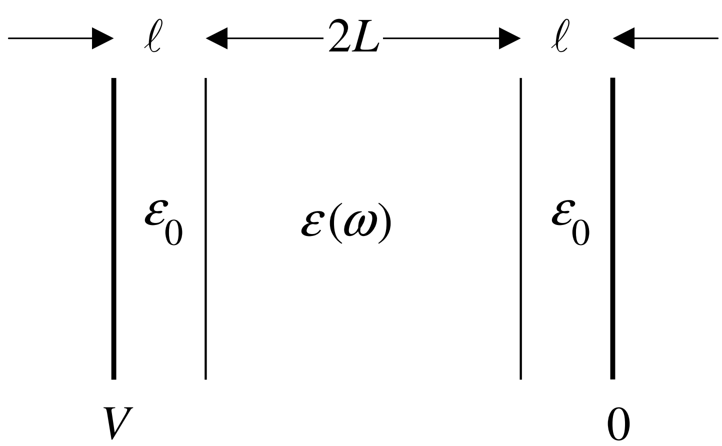
$$\epsilon(\omega)/\epsilon_0 = 1 - \omega_p^2/\omega^2,$$

where ω_p is a constant. In a typical laboratory or astrophysical environment, any attempt to establish a voltage $V(t) = V\cos(\omega t)$ across the plasma generates a region of vacuum (called the "sheath") on either side of the plasma volume as indicated in the one-dimensional sketch below.

(a) Derive expressions for the uniform electric field $E_P(t) = E_P\cos(\omega t)$ in the plasma and for the uniform electric field $E_S(t) = E_S\cos(\omega t)$ in the sheath. Assume there is no free charge anywhere. Assume also that ω_P is small enough that an *electrostatic* approximation is always valid.

(b) Plot the fields E_P and E_S on the same graph as a function of frequency. Discuss *where* the voltage drop occurs and *why* when (1) $\omega \ll \omega_P$, (2) $\omega = \omega_P$, and (3) $\omega \gg \omega_P$.

(c) Make an "LC circuit" interpretation of the resonant behavior at $\omega = \omega_p/\sqrt{1 + L/\ell}$.



Radiation Interference.

Let the origin of coordinates be centered on a compact, time-harmonic, source of electromagnetic radiation. The time-averaged power radiated into a differential element of solid angle $d\Omega$ centered on an observation point $\mathbf{r}$ has the form

$$\frac{dP}{d\Omega} \propto |\hat{\mathbf{r}} \times \boldsymbol{\alpha}| \ .$$

If the source has a time-dependent electric dipole moment $\mathbf{p}(t) = \mathbf{p}_0 \cos \omega t$, $\boldsymbol{\alpha} = \mathbf{p}_0$ and electric dipole radiation is observed at $\mathbf{r}$. If the source has a time-dependent magnetic dipole moment $\mathbf{m}(t) = \mathbf{m}_0 \cos \omega t$, $\boldsymbol{\alpha} = \mathbf{m}_0 \times \hat{\mathbf{r}}$ and magnetic dipole radiation is observed at $\mathbf{r}$. For this problem, consider a source where $\mathbf{p}(t)$ and $\mathbf{m}(t)$ are present simultaneously.

(a) Show that the time-averaged angular distribution of power generally exhibits interference between the electric and magnetic dipole radiations. Under what conditions is there no interference?

(b) Show that the time-averaged *total* power produced by the source does *not* exhibit interference.

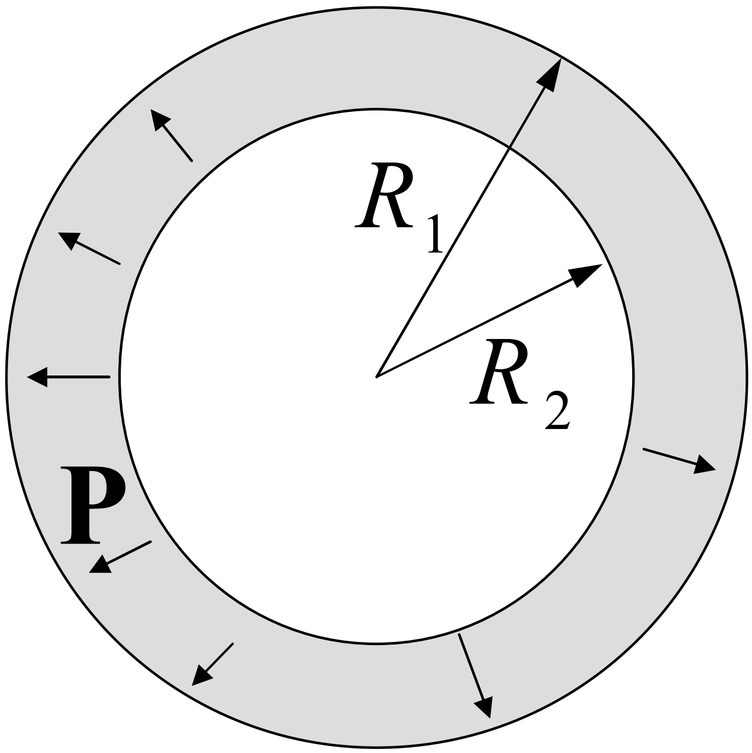
Wave propagation.

The electric field of a wave propagating in a vacuum ($\epsilon = \epsilon_0, \mu = \mu_0$) is $\mathbf{E} = E_0\hat{\mathbf{y}} \exp\left[i(hz - \omega t) - \kappa x\right]$. Compute the magnetic field $\mathbf{B}$. What is the relation between the parameters h, κ, ω? Under what conditionis the polarization of the magnetic field close to circular?

Permanently polarized shell.

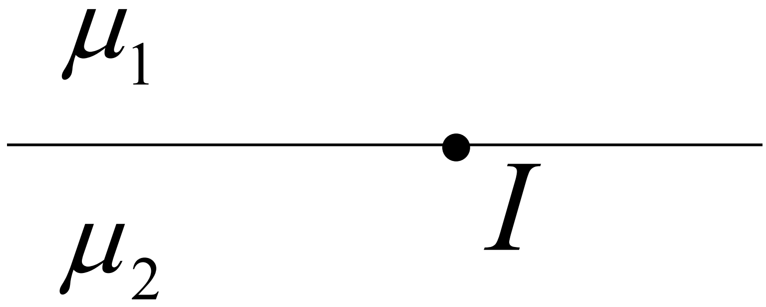
A permanent polarization $\mathbf{P}(\mathbf{r}) = P_0\hat{\mathbf{r}}$ (P_0 is a constant, $\hat{\mathbf{r}} = \mathbf{r}/r$) is maintained inside a spherical shell. The shell is centered at $\mathbf{r} = 0$, its outer and inner radii are R_1 and R_2. Find the electric field everywhere.

Hint: The charge density in the interval $R_2 < r < R_1$ is not zero.



Current in a magnetic medium.

An infinitely long and infinitely thin straight wire carries electric current I. The wire lies at the flat interface between two semi-infinite media with permeabilities μ_1 and μ_2. Find the fields **B** and **H** everywhere.



Physics Comprehensive Exam - Fall 2004

Quantum Mechanics

August 12, 2004
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems in Quantum Mechanics. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

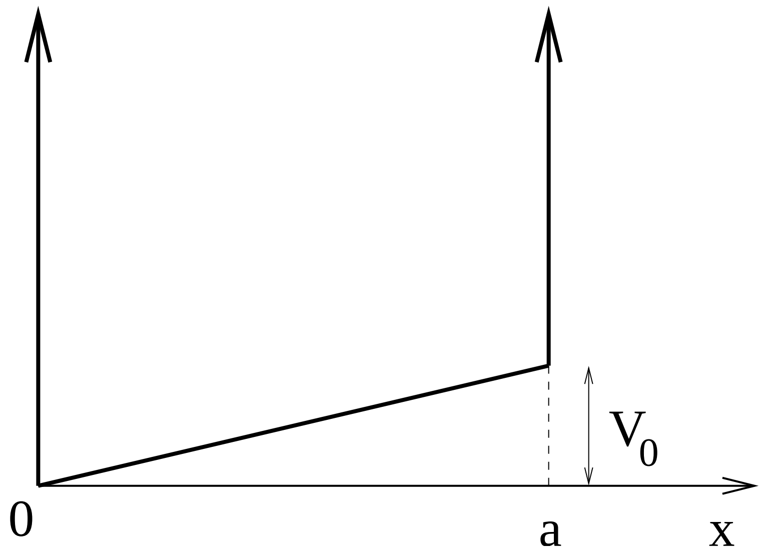
Name: Code Symbol:

Quantum Mechanics problems: 1 2 3 4 5

Perturbation.

Consider the one dimensional potential well of width “a”, with a tilted bottom, as shown below. Use perturbation theory to

a) find the first three energy levels of the system up to the first non-trivial order in V_0;



Electron in a magnetic field.

An electron vith velocity $\mathrm{v} = (\mathrm{v_x}, \mathrm{v_y}, \mathrm{v_z})$ moves in the presence of a magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$, where $B_0 = 1$ Tesla.

(a) Derive an expression for the minimum possible value of the product $\Delta\mathrm{v_x}\Delta\mathrm{v_y}$ of the uncertainties in the components of velocity perpendicular to the field.

(b) Compute numerically its value and demontrate explicitly that the units are correct.

Hints: $\mathbf{p} = m\mathbf{v} + q\mathbf{A}$, and $\Delta\hat{O}_1\Delta\hat{O}_2 \geq \frac{1}{2}|\langle[\hat{O}_1, \hat{O}_2]\rangle|$, where $\hat{O}_1$ and $\hat{O}_2$ are operators.

Potentially useful contants: $h = 6.63 \times 10^{-34}$ J.s; $e = 1.60 \times 10^{-19}$ C; $m = 9.11 \times 10^{-31}$ Kg.

Time dependent perturbation theory.

In magnetic resonance a sample atom sits in a constant magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$ and a small transverse oscillating field $\mathbf{B}_t = b\cos(\omega t)\hat{\mathbf{x}}$. Slightly idealized, the Hamiltonian for an electron spin in the sample is

$$H = -\frac{g\hbar q}{4mc}\left[B_0\sigma_z + b\cos(\omega t)\sigma_x\right] = H_0 + H_1$$

where σ_z, σ_x are Pauli spin matrices: $(\sigma_z)^2 = 1$. Suppose that at time $t = 0$ the spin is prepared with $\sigma_z = +1$. Write the probability $P_-(t)$ for a transition to $\sigma = -1$ at a later time t in first-order perturbation theory. Please use the notation $\omega_0 = (E_- - E_+)/\hbar$.

Angular Momentum.

A perturbation $W = b\hat{L}_x$ is added to an originally unperturbed Hamiltonian $H_0 = a\hat{L}_z$, where the constants a and b have the appropriate units and $b \ll a$. Calculate, to first non-vanishing order in b, the correction to the eigenstates of H_0 introduced by W.

Hint: $\hat{L}_{\pm} = \hat{L}_x \pm i\hat{L}_y$

Observable Algebra.

The two energy eigenstates $|A\rangle$ and $|B\rangle$ form a complete set of orthonormal states for a two-level quantum system. The operator $\hat{R}$ represents an observable for this system.

(a) Find the eigenvalues of $\hat{R}$ if experiments determine that $\langle A|\hat{R}|A\rangle = 1$, $\langle A|\hat{R}^2|A\rangle = 5/4$, and $\langle A|\hat{R}^3|A\rangle = 7/4$.

(b) Find the orthonormal eigenvectors of $\hat{R}$ if $\langle A|\hat{R}|B\rangle = \frac{1}{2}(1 + i)/\sqrt{2}$.

Physics Comprehensive Exam - Fall 2004

Statistical Mechanics & Thermodynamics

August 13, 2004
9:00 am – 2:00 pm

Instructions:

1. Work **three** problems in Statistical Mechanics & **one** problem in Thermodynamics. **DO NOT WORK ALL SIX!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: Code Symbol:

Stat. Mech. problems:	1	2	3	4
Thermo problems:	5	6		

Statistical Mechanics 1

Quantum widgets.

A widget is a device that performs a useful function. An ensemble of N independent quantum widgets is in contact with a thermal reservoir at temperature T. The energy levels of each widget are given by

$$\epsilon_n = \hbar\omega_0 \left(n + \frac{1}{2}\right), \quad \text{with} \quad n = 1, 3, 5, 7, \ldots$$

Note that n is an odd integer.

(a) Obtain an expression for the internal energy u per widget as a function of temperature.

(b) Obtain an expression for the entropy per widget as a function of temperature.

(c) Obtain an expression for the specific heat per widget. What does it reduce to in the classical limit?

Potentially useful relations:

$$\sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \ldots = \frac{1}{1-x};$$

$$\sum_{m=0}^{\infty} m x^m = x \frac{d}{dx} \sum_{m=0}^{\infty} x^m$$

Hydrogen Atom.

The Hydrogen atom energy eigenvalues are given by

$$E_n = -\frac{13.6\ \mathrm{eV}}{n^2}.$$

(a) What is the relative probability to find the atom in the $n = 2$ state compared to the $n = 1$ state? Let $T = 300K$.

(b) Write down the partition function for the hydrogen atom. Formally, the sum diverges. Explain why, in practice, the upper limit of the sum should be cut off at a finite number for the case of a very dilute gas of hydrogen atoms.

Potentially useful constant: $k_B = 8.63 \times 10^{-5}\ \mathrm{eV}/K$

Anharmonic diatomic molecule.

The vibration spectrum of diatomic molecules exhibits anharmonicity when the amplitude of vibration becomes large. The energy levels in the presence of such anharmonicity are approximately given by

$$\varepsilon_n = (n + 1/2)h\nu - x_e(n + 1/2)^2 h\nu$$

for $n = 0, 1, 2, ..,$ where x_e is the parameter that represents the degree of anharmonicity.

(a) Calculate the effect of anharmonicity upon the vibrational specific heat up to first order in x_e.

(b) Interpret your results.

Potentially useful relations:

$$\sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \ldots = \frac{1}{1-x};$$

$$\sum_{m=0}^{\infty} m x^m = x \frac{d}{dx} \sum_{m=0}^{\infty} x^m$$

Classical Rotations.

The rotational spectrum of a diatomic molecule is specified by two angular variables θ, ϕ, and the corresponding canonical conjugate momenta p_θ, p_ϕ. Assume that the form of the kinetic energy of rotational motion is

$$\varepsilon_{rot} = p_\theta^2/2I + p_\phi^2/(2I\sin^2\theta).$$

(a) Derive the classical expression for the rotational partition function.

(b) Calculate the corresponding entropy.

(c) Calculate the specific heat.

(d) Under what conditions do you expect your results to describe well a real diatomic molecule?

Potentially useful integral: $\int_{-\infty}^{\infty} dx \exp(-ax^2) = \sqrt{\pi/a}$

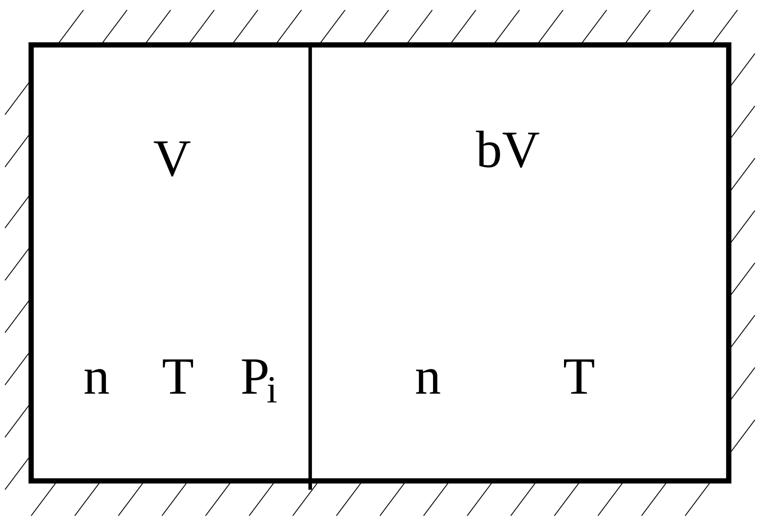
Mixture of ideal gases. A thermally insulated container with a partition is in the initial state shown in the figure below, where $b > 1$, T is the initial temperature. There are n moles of **ideal gas** on each side, and the pressure of the left-hand side is P_i. After the partition is removed,

(a) What is the final pressure in terms of P_i?

(b) What is the entropy change if the two gases are different?

(c) What is the entropy change if the two gases are identical?

Hint: The entropy S of n moles of ideal gas of volume V, at temperature T, is given by $S = nR \ln \left[VT^{3/2}/nN_A \right] + Const.$, where R is the ideal gas constant, and N_A is the Avogadro number.



Thermodynamics 2

One component fluid.

The chemical potential of a one-component fluid with N particles is given by the expression

$$\mu = \mu_0(T) + k_B T \ln\left(\frac{P}{P_0(T)}\right),$$

where T is the temperature, P is the pressure, k_B is the Boltzmann's constant, and the functions $\mu_0(T)$ and $P_0(T)$ are well behaved.

(a) Show that this system obeys Boyle's law, $PV = Nk_B T$.

(b) Obtain an expression for the specific heat at constant pressure.

(c) Obtain the density of Helmholtz free energy, $f(T, V/N)$.

Potentially useful is the Gibbs-Duhem relation: $N d\mu = V dP - S\, dT$.