Physics Comprehensive Exam - Fall 2003

Day 1

Classical Mechanics and Thermodynamics

August 11, 2003
9:00 am – 3:00 pm

Instructions:

1. Do **four** of the five problems in Classical Mechanics and **one** of the two problems in Thermodynamics.
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: Code Symbol:

Classical Mechanics problems:	1	2	3	4	5
Thermodynamics Problems:	1	2			

Classical Mechanics 1

Rigid Body Rotations. In the study of rigid body rotations, the Euler angles (ϕ, θ, ψ) relate the space fixed axes x_0, y_0, z_0 to the body fixed principal axes (x, y, z) by a set of three axis rotations. The first rotation is by an angle ϕ about the z_0 axis to produce the x_1, y_1, z_1 axes. The second rotation is by an angle θ about the x_1 axis to produce the (x_2, y_2, z_2) axes. The third rotation is by an angle ψ about the z_2 axis to produce the (x, y, z) axes.

(a) Draw figures to illustrate these rotations.

- For the remaining questions, consider a symmetric body with the body fixed z-axis as the axis of symmetry:

(b) Write down Euler's equations of motion for a rigid body in terms of the principle moments of inertia $(I_1 = I_2, I_3)$, the angular velocity components $(\omega_x, \omega_y, \omega_z)$, and the torque components (N_x, N_y, N_z).

- In the body fixed axes the angular velocity has components

$$\begin{aligned}\omega_x &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi && (1)\\ \omega_y &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi && (2)\\ \omega_z &= \dot{\phi}\cos\theta + \dot{\psi} && (3)\end{aligned}$$

(Take these as given; do not derive them!)

Now suppose that

$$\begin{aligned}N_x &= N_1\cos\psi + N_2\sin\psi && (4)\\ N_y &= -N_1\sin\psi + N_2\cos\psi && (5)\end{aligned}$$

where N_1 and N_2 are independent of ψ.

(c) Using equations 1, 2, 4, and 5 in your Euler's equations of motion for a rigid body, show that the first two equations may be combined to produce two new equations, both independent of ψ (but not $\dot{\psi}$).
Hint: For conformity, start with $\cos\psi$ times the first equation minus $\sin\psi$ times the second equation. Finally, use equation 3 to replace $\dot{\psi}$ by ω_z. You should arrive at:

$$\begin{aligned}I_1(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta) + I_3\omega_z\dot{\phi}\sin\theta &= N_1 && (6)\\ I_1(\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta) - I_3\omega_z\dot{\theta} &= N_2 && (7)\\ I_3\dot{\omega}_z &= N_z && (8)\end{aligned}$$

(d) What axes are associated with the torques N_1 and N_2?

(e) What are N_1, N_2, and N_z for torques associated with the z_0 axis?

(f) For $N_1 = 0, N_2 = N_0\sin\theta$, and $N_z = 0$ (where N_0 is a constant and with $\dot{\phi}^2$, $\dot{\phi}\dot{\theta}$, $\ddot{\phi}$, and $\ddot{\theta}$ all negligible), solve for $u = \cos\theta$.

Small Amplitude Oscillations of a Rod. Two heavy stationary balls of radius $R = 3.0\ cm$ and mass $M = 1.0\ kg$ are separated by a distance (between their centers) of $2L = 20\ cm$. Two small balls of radius $r = 0.3\ cm$ and mass $m = 10\ g$ are fixed at the ends of an extremely light (but rigid) rod of length $2\ell = 0.1\ m$. The rod is suspended at its center by a a very thin and light thread. In equilibrium, the centers of all balls are aligned, and the center of the rod is equidistant to the centers of the heavy stationary balls.

(a) Write down the equation of motion for the rod.

(b) Calculate the period of small oscillations of the rod.

Sliding Coin. In the figures below, the gravitational acceleration $\mathbf{g}$ points downward into the page.

(a) A uniform thin coin of radius R and mass m slides on its face on a horizontal surface, i.e. its circular axis is vertical and there is no rotation about that axis. The coefficient of friction between the coin and the surface is μ. The initial velocity of the coin is $\mathbf{V}$. What is the instantaneous acceleration $\mathbf{a}$?

(b) The same coin is again set face down on the horizontal surface with its center of mass at rest, but spinning about its axis with angular velocity ω. What is the instantaneous angular acceleration α?

- In parts (c)–(f), the coin is set down with center-of-mass velocity $\mathbf{V}$ (as in part a) *and* angular velocity ω (as in part b). Assuming that there is no tendency for rotation about a horizontal axis (because the coin is thin):

(c) Does the center of mass of the coin move in a straight line or does it deviate to the right or left with respect to $\mathbf{V}$? Why?

(d) Do you expect the instantaneous center of mass acceleration $\mathbf{a}$ to have a magnitude less than, equal to, or greater than the value you found in part (a)? Why?

(e) Do you expect the magnitude of the instantaneous angular acceleration α to be less than, equal to, or greater than the value you found in part (b)? Why?

(f) Find by calculation the instantaneous acceleration $\mathbf{a}$ of the center of mass. Assume for simplicity that $V > \omega R$.

Helical Motion. A frictionless helical wire has a radius R, and is aligned along a vertical axis. The coils of the helix are spaced such that each complete revolution about the z-axis corresponds to a rise/drop of $\Delta z = b$. A glass bead, initially at rest, is allowed to slide down the wire under the influence of gravity. Assume that the motion is frictionless.

(a) Find an appropriate Lagrangian for the constrained motion, and determine the appropriate equation of motion fo the bead.

(b) Determine the force of constraint exerted by the wire on the bead, and verify that this force has no component along the wire itself.

Center of Mass of Solids. Find the center of mass of a solid hemisphere of constant density.

Thermodynamics 1

Thermal Engine. Imagine a thermal engine consisting of 1 kg of water (the heater) and 1 kg of ice (the refrigerator). The initial water temperature is $T_1 = 373\ K$, while the initial ice temperature is $T_2 = 273\ K$. The latent heat associated with ice melting is $L = 335 J/g$. Neglect the temperature depence of the specific heat of water.

(a) What is the work produced immediately after all the ice has melted?

(b) What is the water temperature immediatelly after all the ice has melted?

Thermodynamics 2

Thermodynamic Equilibrium of Two Phases. Consider two phases in equilibrium with each other. For phase I the free energy, entropy and volume are G_I, S_I and V_I, respectively. Similarly for phase II we have G_{II}, S_{II} and V_{II}.

(a) Show that, at the phase boundary,

$$S_{II} - S_I = \frac{H_{II} - H_I}{T}.$$

(b) Assume that phase II is an ideal gas, that its volume $V_{II} \gg V_I$, and calculate the (P,T) phase line in terms of ΔH, P, and T. Assume that at $T = T_0$ then $P = P_0$.

Physics Comprehensive Exam - Fall 2003

Day 2

Electricity & Magnetism and Statistical Mechanics

August 12, 2003
9:00 am – 3:00 pm

Instructions:

1. Do **four** of the five problems in Electricity & Magnetism and **one** of the two problems in Statistical Mechanics.
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: Code Symbol:

Electricity & Magnetism problems:	1	2	3	4	5
Statistical Mechanics problems:	1	2			

Electricity & Magnetism 1

A Point Charge and an Ideal Dieletric. An ideal dielectric with permittivity ϵ fills the half-space $z < 0$. A point charge Q is placed at $z = a$. Find the electric field $\mathbf{E}$ everywhere.

Electricity & Magnetism 2

Magnetic Insulating Disk. A disk of radius R and thickness $h \ll R$ is made of insulating magnetic material with permeability μ. The disk is placed into a uniform magnetic induction field $\mathbf{B}_0$ directed along its axis. If the disk were infinitely thin ($h \to 0$) then the magnetic induction $\mathbf{B}$ in the center of the disk would obviously coincide with $\mathbf{B}_0$. Find $\Delta\mathbf{B} = \mathbf{B} - \mathbf{B}_0$ to leading order in h/R.

Electricity & Magnetism 3

Charge Density Relaxation. In a uniform medium with conductivity σ and dielectric permittivity ϵ external forces maintain a certain static distribution of charge $\rho_0(\mathbf{r})$ which induces the electric field $\mathbf{E}_0(\mathbf{r})$. At $t = 0$ the external forces are switched off.

(a) Determine the law of relaxation of the charge density $\rho(\mathbf{r}, t)$.

(b) Determine the electric field $\mathbf{E}(\mathbf{r}, t)$.

(c) Find the magnetic field $\mathbf{H}(\mathbf{r}, t)$.

Snell's Law for Electrons. Let $V_< =$ constant be the electrostatic potential of the half-space $z < 0$ and let $V_> =$ constant be the electrostatic potential of the half-space $z > 0$.

(a) The trajectory of an electron that passes through $z = 0$ consists of two straight lines which make angles $\theta_<$ and $\theta_>$ with respect to the normal to the interface. Use energy conservation to show that

$$\frac{\sin\theta_>}{\sin\theta_<} = \sqrt{\frac{V_<}{V_>}}$$

where θ is the angle between the trajectory and the normal to the interface.

(b) What type of charge distribution must be present at the interface ($z = 0$) to create this physical situation?

Magnetic Field Inside Superconductors. In 1935, the brothers F. and H. London proposed to describe superconductivity by the following phenomenological constitutive relations

$$\frac{\partial}{\partial t}(\Lambda \mathbf{J}) = \mathbf{E}, \quad \nabla \times (\Lambda \mathbf{J}) = -\mathbf{B},$$

where Λ is a constant. Maxwell's equations as well as the corresponding boundary conditions remain unchanged. Consider an infinite superconducting slab of thickness $2d$ placed in a constant magnetic field $\mathbf{H}_0$ parallel to the surface. Assuming the validity of London's equations, compute $\mathbf{H}$ and $\mathbf{J}$ inside the slab.

Statistical Mechanics 1

Solid Hydrogen. Solid Hydrogen is a dieletric at normal pressure and its density is $76\,kg/m^3$. To become metallic, its Fermi energy needs to equal to its atomic ionization potential $E_{ion} = 13.6eV$. Give numerical estimates for the questions below.

(a) What is the pressure required to obtain this metallic transition?

(b) What is the density corresponding to this pressure?

Statistical Mechanics 2

Rotating Ideal Gas. Consider an ideal gas at temperature T, inside a cylinder of radius R and height L. Assume that the number of gas particles is N and that the cylinder rotates around its axis at a constant angular velocity ω.

(a) Determine the density distribution of particles inside the cylinder.

(b) Determine the pressure the gas exerts on the cylinder wall.

Physics Comprehensive Exam - Fall 2003

Day 3

Quantum Mechanics
and
Statistical Mechanics/Thermodynamics

August 13, 2003
9:00 am – 3:00 pm

Instructions:

1. Do **four** of the five problems in Quantum Mechanics and **one** of the two problems in Statistical Mechanics/Thermodynamics.
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
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GOOD LUCK!

Name: Code Symbol:

Quantum Mechanics problems:	1	2	3	4	5
Statistical Mechanics problem:	3				
Thermodynamics problem:	3				

Ramsauer-Townsend Effect in One Dimension. A beam of particles is scattered by a one-dimensional square well. The particles have mass $1.675 \times 10^{-27}\ kg$. Transmission maxima are observed for beam energies of 1.150, 23.656, and $50.254\ MeV$

(a) What is the width of the potential in fermis (1 fermi = $10^{-15}\ m$) ?

(b) What is the depth of the potential in MeV ?

Time Dependent Interaction and the Harmonic Oscillator. Consider a quantum harmonic oscillator with Hamiltonian

$$H = \hbar\omega_0(a^\dagger a + 1/2)$$

and eigenstates $|n\rangle$, where

$$H|n\rangle = \hbar\omega_0(n + 1/2)|n\rangle.$$

Add a time dependent interaction Hamiltonian of the form

$$V(t) = F(t)|1\rangle\langle 0| + F^*(t)|0\rangle\langle 1|.$$

(a) Let a general solution be expressed as

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n(t)|n\rangle$$

and determine the evolution equations for all of the $c_n(t)$'s.

(b) If $F(t) = \sqrt{2}\hbar\omega_0\Theta(t)$, where $\Theta(t)$ is the Heaviside function, i.e., $\Theta(t) = 1$ for $t \geq 0$ and $\Theta(t) = 0$ for $t < 0$, obtain the energy eigenvalues and eigenstates.

(c) If $|\psi(0)\rangle = |0\rangle$ at $t = 0$, find $c_n(t)$ for all $t > 0$ and $n \geq 0$.

Exotic Harmonic Oscillator. The point of suspension x_0 (spatial origin of the potential) of a harmonic oscillator start to move at time $t = 0$, when the system is in its ground state. The point of suspension moves according to the law $x_0 = x_0(t)$. At $t = T$, the point of suspension is fixed again.

(a) Find the expectation values for the position and momentum operators as a function of time for any time $t > 0$.

(b) Find the probability of excitation of the first excited state (due to the motion of the point of suspension) assuming that perturbation theory is applicable.

Stark Effect in a Two-dimensional Rotor. A rigid charged rotor with moment of inertia I and electric dipole moment $\mathbf{p}$ is rotating in a plane. A small homogeneous electric field $\mathbf{E}$ is applied along the x direction. Use perturbation theory to calculate the first non-trivial correction to the energy of **each** eigenstate of the quantum rotor.

Spin-Orbit Coupling in a Central Potential. A particle moves in a central potential $V(r)$. The Hamiltonian of the particle is

$$H = H_0 + \frac{C}{\hbar^2}\mathbf{L}\cdot\mathbf{S} - \frac{B_z}{\hbar}(L_z + 2S_z),$$

where $\mathbf{L}$ is the orbital angular momentum operator, $\mathbf{S}$ is the intrinsic spin operator for the particle and

$$H_0 = \frac{p^2}{2m} + V(r).$$

1. If $C/\hbar^2 \gg B_z/\hbar$, which quantum numbers are good? (Ignore the radial quantum number n).
2. If $C/\hbar^2 \ll B_z/\hbar$, which quantum numbers are good? (Ignore the radial quantum number n).
3. For case 1 obtain a general expression for the approximate eigenvalues of H. Take the eigenvalues of H_0 to be E_n. If you use perturbation theory, do not go higher than second order.
4. For case 2 obtain a general expression for the approximate eigenvalues of H. Take the eigenvalues of H_0 to be E_n. If you use perturbation theory, do not go higher than second order.

Statistical Mechanics 3

Do Fermion Pairs Behave Like Bosons?. Let $a_1, a_1^\dagger, a_2, a_2^\dagger$ be fermi annihilation and creation operators for two different states, 1 and 2. Sometimes people say that a pair of fermions behaves like a boson. Define $B^\dagger = a_1^\dagger a_2^\dagger$, and $B = a_2 a_1$. Using the anti-commutation rules for the a's, find the commutation rule for $[B^\dagger, B] =$? Express your answer solely in terms of $n_1 = a_1^\dagger a_1$ and $n_2 = a_2^\dagger a_2$. Discuss this result as $T \to 0$.

Thermodynamics 3

Cooling Copper Ball. Calculate the time required for a copper ball to cool in vaccum from $T_1 = 500\ K$ to $T_2 = 300\ K$. The ball radius is $1\ cm$, the absorption coeficient of its surface is $\epsilon = 0.8$, the specific heat is $C = 390\ J/kg/K$, and its density $\rho = 8.9 \times 10^3\ kg/m^3$.