Comprehensive Exam — Fall 2000

Classical Mechanics

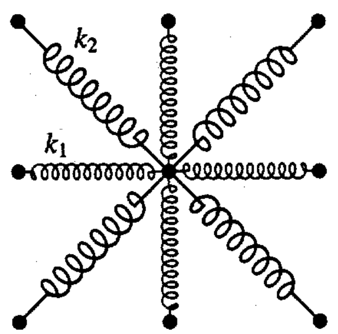
1. Determine the Hamiltonian of an anharmonic oscillator, if the Lagrangian is given by

$$\mathcal{L} = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2 x^2 - \alpha x^3 + \beta x \dot{x}^2.$$

2. Consider an infinite monoatomic crystal with a simple cubic lattice. Classically such a crystal can be represented as a collection of point masses m connected by springs with constants k.

   (a) Find the normal modes (phonons) and normal frequencies (dispersion relation) of the crystallic lattice assuming that springs only connect nearest neighbors.

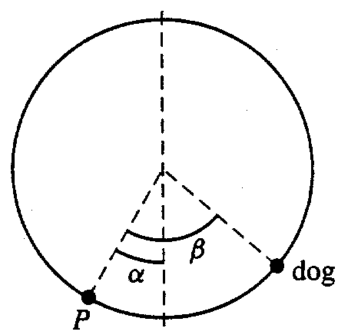
   (b) Derive the equations for normal modes and normal frequencies assuming that in addition to springs with constants k_1 connecting nearest neighbors there are springs with constants k_2 connecting next nearest neighbors. Solve these equations for the special case when the atoms are only displaced along one (say, x) direction.



3. A regular tetrahedron is made of four unit masses joined by massless rigid rods of unit length. What are its principal moments of inertia relative to the center of mass?

4. A horizontal circular disk of mass M is free to rotate about a vertical axis through a point P on its rim. If a dog of mass m walks once around the rim, show that the disk turns through an angle given by the expression

$$\int_0^{\pi} \frac{4m\cos^2\gamma \, d\gamma}{(3M/2) + 4m\cos^2\gamma}.$$

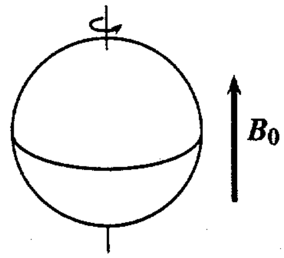


5. A rocket is shot out from the earth into interstellar space. Except for a short time in the beginning, the acceleration a' of the rocket, as measured by the passengers, is constant. The rocket has been aimed at a star a fixed distance D from earth, and moves on a straight line. According to clocks inside the rocket, how long will it take to get to the star?

   Express your answer in terms of the constant distance (D) and acceleration (a').

Electricity and Magnetism

1. A perfectly conducting spherical shell of radius R is rotating about the z-axis with angular velocity ω in a uniform magnetic field B_0 along the z- direction. What is the emf developed between the equator and the north pole?



2. Two unequal capacitors (C_1, C_2) are charged separately to have the same potential difference V. Subsequently, the positive terminal of one is connected to the negative terminal of the other. Then the other two terminals are shorted (connected together).

   (a) Calculate the loss of electrostatic energy.

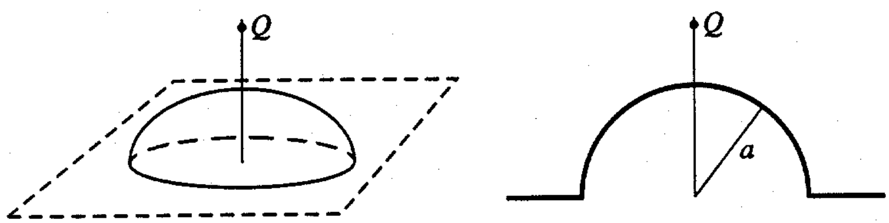
   A capacitor (C) is charged by a battery (potential V) through a resistor R.

   b) Calculate the energy dissipated after the capacitor is fully charged.

   c) Calculate the energy supplied by the battery to fully charge the capacitor.

3. A cubical resonant cavity has perfectly conducting walls at $x = 0, x = a, y = 0, y = a, z = 0$, and $z = a$. The region $0 \le z \le b\,(b < a)$ is filled with a non-magnetic dielectric with permittivity ϵ. Derive a transcendental equation for the frequencies of the cavity normal modes which have $E_y = E_z = 0$. Solve this equation when $\epsilon = 1$.

4. A conductor at potential $V = 0$ has the shape of an infinite plane except for a hemispherical bulge of radius a. A charge Q is placed above the center of the bulge, a distance p from the plane (or $p - a$ from the top of the bulge). What is the force on the charge?



5. At time $t = 0$, an electron orbits a classical hydrogen atom at a radius a_0 equal to the first Bohr radius. Assuming the energy loss per revolution to be small compared to the total energy of the atom, derive an expression for the time it takes for the radius to decrease to zero due to radiation. Estimate, in sec, the lifetime of such a "classical atom".

$$P = \tfrac{2}{3}e^2a^2/c^3$$

Quantum Mechanics

1. A system consists of two identical non-interacting particles of mass m, confined within a one-dimensional box $x \in [0, L]$. Denote by E_0 the ground state energy, and by E_1 the first excited energy.

   (a) Assuming the particles have spin zero, write down the system energy eigenfunctions $\psi(x_1, x_2)$ for energy E_0 and for energy E_1. Make sure the wave functions are properly normalized and symmetrized.

   (b) Find the mass density $\rho(x)\,\mathrm{d}x$ for the system in the ground state. (As a check, your answer should satisfy $\int \rho(x)\,\mathrm{d}x = 2m$.)

   (c) Repeat part (a), assuming instead that the particles have spin-1/2.

2. Given a quantum mechanical system with a singular potential:

$$V(x) = -aV_0\delta(x).$$

   Show that the boundary conditions are

$$\text{a)}\quad \phi(0^-) = \phi(0^+),$$

$$\text{b)}\quad \frac{\hbar^2}{2m}\left(\phi'(0^+) - \phi'(0^-)\right) = -aV_0\phi(0).$$

   Find the eigenfunction for a bound state of this system. How many bound states are there?

3. Given a system of spin-1/2 particles, and denote by $\sigma_x^{(i)}, \sigma_y^{(i)}, \sigma_z^{(i)}$ the usual Pauli spin operators for the i-th particle.

   (a) Assume a coupling (single particle interaction)

$$V_i(t) = -\frac{\hbar\Omega_R}{2}\Big(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\Big) = -\frac{\hbar\Omega_R}{2}\left(\sigma_+^{(i)} + \sigma_-^{(i)}\right) = -\hbar\Omega_R\sigma_x^{(i)},$$

   with $\sigma_\pm = \sigma_x \pm \mathrm{i}\,\sigma_y$. Find

$$U(t) = \mathrm{e}^{-\mathrm{i}V(t)t/\hbar} = ?$$

(b) For the interaction (2-particle entanglement)

$$V_{[x]}(t) = -\frac{\hbar\Omega_R}{2}\left(\sigma_x^{(1)}\sigma_x^{(2)}\right),$$

find

$$U(t) = e^{-iV_{[x]}t/\hbar}$$

at $\Omega_R t = \pi/2$.

4. A linear harmonic oscillator is subject to the action of a uniform electric field, which is regarded as a perturbation and which changes in time according to the equation

$$\mathcal{E}(t) = A\,\frac{1}{\tau\sqrt{\pi}}\,e^{-(t/\tau)^2},$$

where A is a constant.

Considering that the oscillator was in the ground state until the field was switched on (at $t = -\infty$), compute in the first approximation the probability of its excitation as a result of the action of the above perturbation (as $t \to \infty$).

5. For deuterium (H^2) and tritium (H^3), compute numerically (in Å) the wave length of the H_α-line of the Balmer series, given the wave length $\lambda_\alpha = 6562.79$ Å for hydrogen (H^1). $m_p/m_e \approx m_n/m_e \approx 1838$

Thermodynamics

1. (a) Consider a PVT system whose equation of state has the form $P = f(V)T$. Show that the internal energy U is independent of V.

   Here P is the pressure, T the absolute temperature, and $f(V)$ is a function of volume V only. It is **not** sufficient to demonstrate this for a specific PVT system, e.g., the ideal gas.

   (b) Consider a PVT system whose equation of state has the form $P = u(T)/3$, where u is the internal energy per unit volume, $u = U/V$. Determine the functional form of $u(T)$.

2. Suppose you use your furnace to increase the temperature of your house from 18°C to 23°C. Does the *energy* of the room increase?

3. A block of boron is placed in contact with a block of solid neon. The blocks are isolated and no chemical reaction occurs. The following data apply:

   Boron $N_B = 1.21 \times 10^{24}$, initial temperature $T_B = 120\,\mathrm{K}$,

   Neon $N_{Ne} = 1.0 \times 10^{23}$, initial temperature $T_{Ne} = 90\,\mathrm{K}$.

   The Debye temperatures are $\Theta_{D,Ne} = 75\,\mathrm{K}$ for Neon, $\Theta_{D,B} = 1250\,\mathrm{K}$ for Boron.

   $C_{Ne} = 3N_{Ne}k$ for sufficiently high T, $C_B = \frac{12\pi^4}{5}Nk\left(\frac{T}{\Theta_D}\right)^3$ for sufficiently low T.

   Find the final temperature T_f.

Statistical Mechanics

1. Consider a system of three-dimensional rotators (with two degrees of freedom and no translational motion) in thermal equilibrium according to Boltzmann statistics; take account of the quantization of energy. Calculate the free energy, entropy, energy and heat capacity (per rotator) in the case of high temperatures, making use of Euler's approximation formula:

$$\sum_{J=0} f(J + \tfrac{1}{2}) = \int_0^\infty f(x)\,\mathrm{d}x + \frac{1}{24}[f'(0) - f'(\infty)] + \ldots$$

2. Three particles at the corners of an equilateral triangle (see figure) each carry a quantum mechanical spin 1/2, their mutual spin Hamiltonian being given by

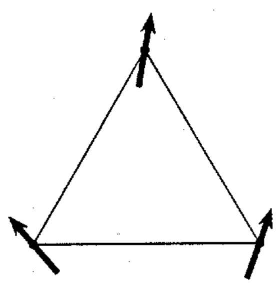
$$H = \frac{\lambda}{3}\left(\sigma_1 \cdot \sigma_2 + \sigma_1 \cdot \sigma_3 + \sigma_2 \cdot \sigma_3\right).$$

   (a) List the energy levels of this spin system, giving their total spin values and degeneracies.

   (b) Deduce the partition function Z.

   (c) Calculate the free energy, entropy, and specific heat and sketch their graphs.

   (d) Discuss the physical limits of small and high temperatures for the quantities calculated in b) and c), and interpret your results physically.



3. Consider an ideal gas in two dimensions, consisting of N atoms, each of mass M, confined to an area $A = L^2$, in equilibrium at a temperature T. Each atom has two internal energy states, one with an energy $-\epsilon$ and the other with energy ϵ.

   (a) Find an expression for the pressure as a function of T, A, and N.

   (b) Find an expression for the thermodynamic energy $U(T, A, N)$. Also, find a simpler expression for U valid in the high temperature limit. (Note: the answer $U = \infty$ is too crude an approximation.)