Physics Comprehensive Exam - Winter 2007

Electricity & Magnetism

November 5, 2007
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems in Electricity & Magnetism. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: Code Symbol:

Electricity & Magnetism problems: 1 2 3 4 5

Electricity and Magnetism 1

Magnetic Field of an AC Capacitor.

A voltage $V(t) = V_0 \sin \omega t$ is applied between the circular plates of a capacitor filled with Ohmic matter of conductivity σ. The radius R of the plates is very large compared to the plate separation d. Find the magnetic field between the plates in the quasi-electrostatic approximation (neglect Faraday induction).

A Hole in Radially Polarized Matter.

The polarization in all of space has the form

$$\mathbf{P} = \begin{cases} 0, & r \leq R \\ \\ P\hat{\mathbf{r}}, & r \geq R. \end{cases}$$

Find

(a) the volume polarization charge density;

(b) the surface polarization charge density; and

(c) the electric field everywhere.

Electricity and Magnetism 3

No Reflection.

A plane wave in vacuum (ϵ_0, μ_0) impinges at normal incidence onto the planar surface of a semi-infinite piece of matter (ϵ, μ).

(a) Assume a plane wave solution and use the Maxwell equations explicitly to derive the relation between wave vector $\mathbf{k}$ and the frequency ω in the medium.

(b) Apply the interface matching conditions explicitly and determine the conditions when no reflected wave appears.

Electricity and Magnetism 4

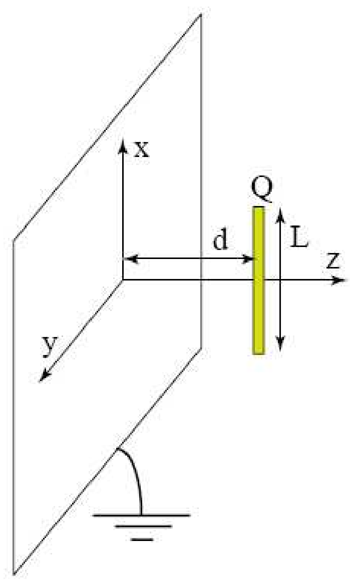
Induced Charge.

A rod of length L and net charge Q (distributed uniformly over its length), is placed parallel to the grounded infinite conducting plane at the distance d from the plane (see the Figure).

(1) Set up an integral expression for the force on the rod. Do not evaluate the integral.

(2) Find the force on the rod in the limit of $d \gg L$ by any method.

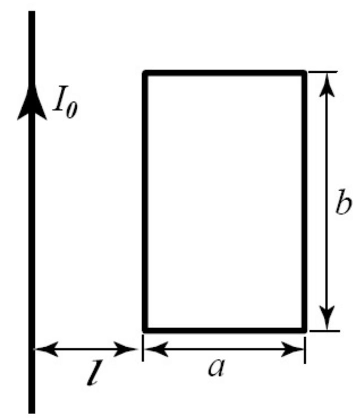
(3) Find the force on the rod in the limit of $d \ll L$ by any method.



Wire Frame.

A rectangular conducting wire frame with sides a and b is at rest on a frictionless horizontal surface at the distance l from the straight long wire carrying current I_0 (see the Figure). The mass of the frame is m, and its resistance is R. Find the magnitude and the direction of the velocity of the frame after the current in the long straight wire has been abruptly switched off. Assume that the current has been switched off very quickly such that the frame was still at rest as it happened.

Hint: Find the net force on the frame and consider its impulse.



Physics Comprehensive Exam - Winter 2007

Quantum Mechanics

November 6, 2007
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems problems in Quantum Mechanics. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: Code Symbol:

Quantum Mechanics problems: 1 2 3 4 5

Quantum Mechanics 1

Atomic Transition.

Using first order time-dependent perturbation theory, determine the transition rate for a hydrogen atom to go from the $|n, l, m\rangle$ state $|2, 1, 0\rangle$ to the ground state $|1, 0, 0\rangle$ under the perturbation $\Delta V = eE_0 z e^{i\omega t}$, where $\omega = E_R/\hbar$ and $E_R = 13.6$ eV is the Rydberg energy.

Hint: You may find the following data useful

$$\int_0^\infty r^n e^{-\mu r} dr = \frac{n!}{\mu^{n+1}}$$

$$|1, 0, 0\rangle = \frac{a_0^{-3/2}}{\sqrt{\pi}} e^{-r/a_0}$$

$$|2, 1, 0\rangle = \frac{(2a_0)^{-3/2}}{\sqrt{4\pi}} \frac{r}{a_0} e^{-r/a_0} \cos\theta$$

Quantum Mechanics 2

Quantum Resonator.

Consider a quantum particle confined inside a spherical shell:

$$V(r) = \begin{cases} 0, & a < r < b \\ \infty, & 0 < r < a \text{ or } r > b \end{cases}$$

Find the wave functions and energies corresponding to the s-states (ones with zero angular momentum).

Hint: The Laplacian in spherical coordinates is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)$$

Quantum Mechanics 3

Angular Momentum.

Consider a quantum particle whose dynamics is described by the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(r).$$

Find the value of the commutator $[H, L_x]$, where L_x is the operator describing the x-component of the angular momentum.

(a) Give the answer using a physical argument without doing any explicit calculations.

(b) Prove the result you gave in part (a) via direct calculation.

Quantum Mechanics 4

Electron in a Uniform Magnetic Field.

An elecron in free space moves under the influence of a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. Neglect electron spin. The Hamiltonian for an electron under magnetic field is given by

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2,$$

where $\mathbf{A}$ is the vector potential. Choose a gauge where $A_z = 0$.

(a) Prove the commutation relationship $[Q, P] = i\hbar$, if we define the canonically conjugate variables Q and P as

$$Q = \left(\frac{c}{eB}\right)^{\frac{1}{2}}\left(p_x - \frac{e}{c}A_x\right) \quad \text{and} \quad P = \left(\frac{c}{eB}\right)^{\frac{1}{2}}\left(p_y - \frac{e}{c}A_y\right).$$

(b) Using part (a) determine the energy levels for the electrons.

Quantum Mechanics 5

Commutator Algebra.

A particle of mass M in a potential $V(x)$ has a bound ground state $|0\rangle$.

(a) Prove the double commutator

$$[x, [H, x]] = \frac{\hbar^2}{M}.$$

(b) By expanding the double commutator in (a), show that

$$\sum_m |\langle 0|x|m\rangle|^2 (E_m - E_0) = \frac{\hbar^2}{2M},$$

where the sum is over a complete set of excited states $|m\rangle$.

Physics Comprehensive Exam - Winter 2007

Statistical Mechanics & Thermodynamics

November 7, 2007
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems problems in Statistical Mechanics & Thermodynamics. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: Code Symbol:

Stat. Mech. & Thermo problems: 1 2 3 4 5

Statistical Mechanics and Thermodynamics 1

State of the Atmosphere.

Suppose the atmosphere is an ideal gas with molar mass m.

(a) Assume near-Earth gravity and show that the atmospheric pressure p varies with height z as

$$\frac{dp}{p} = -\frac{mg}{RT}dz,$$

where R is the universal gas constant.

(b) Suppose the pressure decrease with height is due to adiabatic expansion. Show that

$$\frac{dp}{p} = -\frac{\gamma}{\gamma - 1}\frac{dT}{T},$$

where $\gamma = C_p/C_v$.

(c) Evaluate the change in temperature with height for a pure N_2 atmosphere with $\gamma = 1.4$.

(d) Suppose the temperature is constant throughout the atmosphere. Find $p(z)$ in terms of T and the sea level pressure $p_0 = p(z = 0)$.

(e) The Clausius-Clapeyron equation for the change of boiling temperature T_b as function of pressure can be written as

$$\frac{dp}{dT_b} = \frac{\alpha}{T_b},$$

where $\alpha = 1.4 \cdot 10^6$ J/m^3 for water. Use this equation to estimate the change of the boiling temperature of water with height near sea level. The atmosphere temperature is assumed constant. The density of the atmosphere near sea level is $\rho = 1.29$ kg/m^3. The boiling temperature of water near sea level is $T_b = 100$ °C.

Hint: You can solve (c), (d) and (e) without deriving the formulas in (a) and (b).

Statistical Mechanics and Thermodynamics 2

Heat Pump Efficiency.

What is the shortest time needed to cool down a room from 30 °C to 20 °C using an air conditioning system employing a heat pump with an output power of 1000 W? The outside temperature is 40 °C. The room is assumed adiabatic with the area of 800 ft^2 and the height of 8 ft. The molar volume of air is 22.4 liters. The air contains mostly diatomic molecules, so its specific heat is $C_v = 5R/2$ with R – the universal gas constant.

Hint: Given the work ΔW done by the heat pump and the heat ΔQ removed by it, the maximum efficiency is given by the Carnot theorem:

$$\eta_{max} = \frac{\Delta Q}{\Delta W} = \frac{T_{inside}}{T_{outside} - T_{inside}}.$$

Dilute Diatomic Gas.

Consider a dilute diatomic gas with non-identical pairs of atoms. The energy levels of each molecule are

$$E_j = \frac{\hbar^2}{2I} j(j+1), \qquad j = 0, 1, 2, \cdots,$$

where I is the molecules' moment of inertia. Each energy level is $(2j+1)$-fold degenerate.

(a) Obtain an expression for the rotational contribution to the specific heat C of the gas. (You don't need to perform any derivatives or sums explicitly.)

(b) Derive an approximate expression for C at very low temperature.

(c) Starting from part (a), show that $C \to Nk_B$ at high temperature.

Note: make the calculations sufficiently exact to obtain the lowest order non-zero contribution.

Statistical Mechanics and Thermodynamics 4

White Dwarf.

This problem concerns some thermodynamical properties of a *white dwarf*. Such a star will be supposed to have a temperature $T = 10^7$ K, a mass of $M = 10^{30}$ kg and a radius $R = 5,000$ km. Because of the temperature, the matter in the star is entirely ionized and will be seen as a plasma of nuclei and electrons. Assume the nuclei have an average atomic mass $A = 15$ and include $Z = 7$ protons.

(a) Compute the density of nuclei and of electrons, namely their number per unit volume. Compute the average distance between two electrons.

(b) Compute the Fermi temperature Θ_F of the electron gas, assuming that electrons are not relativistic (it will also be assumed that the Coulomb potential is screened by the motion of the charges). Compare it to the temperature of the star. Can this gas be considered as classical rather than quantum?

You may need these fundamental constants:

- Boltzmann's constant: $k_B = 1.3807 \times 10^{-23}$ J/K,

- Planck's constant: $\hbar = h/2\pi = 1.055 \times 10^{-34}$ Js,

- proton mass: $m_p = 1.67 \times 10^{-27}$ kg,

- electron mass: $m_e = 9.1095 \times 10^{-31}$ kg.

Intermediate Statistics.

Consider an ideal gas in which the maximum number of particles in any one energy state is ρ. Derive the mean occupation number $\langle n_s \rangle$ in the energy state with energy ϵ_s (as a function of the temperature and the chemical potential). Draw the curve giving $\langle n_s \rangle$ as a function of the energy ϵ_s. Show that the Fermi and Bose formulæ follow as special case.

Hint: You may find the following summation formula useful:

$$e^{nx} - 1 = (e^x - 1) \sum_{j=1}^{n-1} e^{jx}$$

Physics Comprehensive Exam - Winter 2007

Classical Mechanics

November 8, 2007
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems in Classical Mechanics. **DO NOT WORK ALL FIVE!**
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GOOD LUCK!

Name: Code Symbol:

Classical Mechanics problems: 1 2 3 4 5

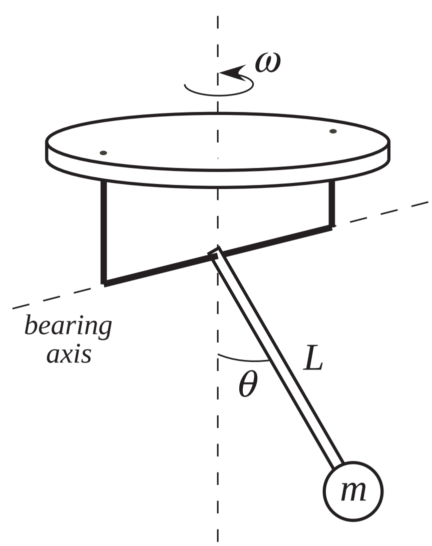
Spinning Pendulum.

A pendulum consists of a rigid massless rod of length L, with a bob of mass m attached at one end. The rod is attached to a bearing that constrains the pendulum to rotate in a plane perpendicular to the bearing axis. The bearing itself is forced to rotate about the vertical axis with a constant angular speed ω. Neglect the kinetic energy of the bearing.

(a) Find expressions for the potential and kinetic energies of the system, and determine the Euler-Lagrange equation of motion for the angular coordinate θ.

(b) Show that for angular speeds below a critical value ω_c, the position $\theta = 0$ is a stable equilibrium point for the pendulum motion. Find an expression for the angular speed ω_c at which this equilibrium become *unstable*. What is the frequency of small amplitude oscillations about $\theta = 0$?

(c) For speeds ω above the critical value, is there an non-zero equilibrium position for the pendulum? If it exists, is it a *stable* equilibrium?



Classical Mechanics 2

Bar on Springs.

A uniform (rigid) bar of length L and mass M is supported at each end by identical springs with elastic constant k, that are constrained to stretch or compress only in the vertical direction. At equilibrium, the bar is exactly horizontal, with the springs *slightly* compressed, so that $2\,k\Delta y = Mg$.

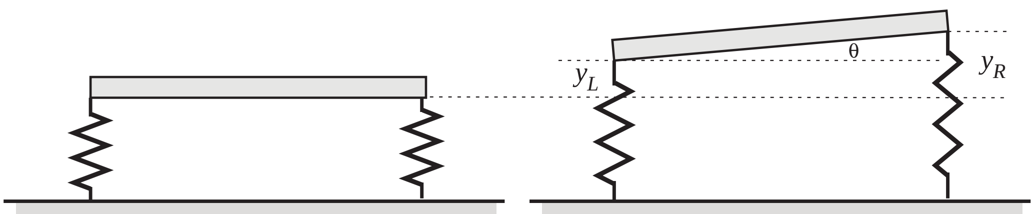
(a) Find expressions for the matrices $\mathcal{M}$ and $\mathcal{V}$ that will allow the kinetic and potential energy of the bar to be written in the forms:

$$T = \frac{1}{2}\sum_{\alpha,\beta} \mathcal{M}_{\alpha\beta}\, \dot{y}_\alpha\, \dot{y}_\beta \qquad V = \frac{1}{2}\sum_{\alpha,\beta} \mathcal{V}_{\alpha\beta}\, y_\alpha\, y_\beta$$

Note: the moment of inertia of a solid bar about its center of mass is $\mathcal{I}_{cm} = \frac{1}{12}ML^2$.

(b) Identify the normal modes of small-amplitude oscillation for the bar. For each mode, specify the frequency oscillation and describe the overall motion of the bar, *as a whole*.

(c) Suppose that one end of the bar is given a small downward displacement d. Describe the subsequent motion of the bar.



Classical Mechanics 3

Rocket in Orbit.

A rocket of mass m is in a circular orbit around the Earth, of radius r_o.

(a) Find expressions for the speed, total energy, and angular momentum for the rocket, in terms of the orbit radius (along with any other relevant physics quantities).

(b) Suppose that the rocket engine ignites for a quick "burn", suddenly increasing the rocket's speed by 25 percent. What will be the apogee (i.e. maximal distance) of the rocket's new orbit?

Classical Mechanics 4

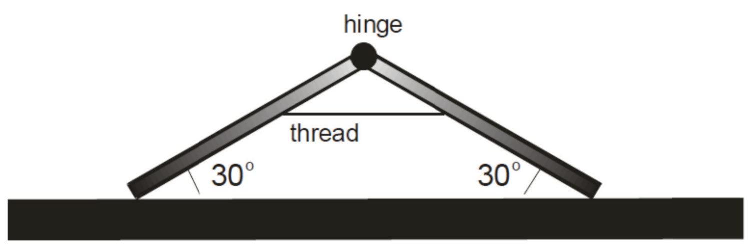
Falling Hinge.

Two thin homogeneous beams, each of mass m and length l, are connected by a frictionless hinge and a thread. In the following you may neglect the thread and the mass of the hinge.The system rests on a smooth surface in the way shown in the Figure. At $t = 0$ the thread is cut.

(a) Find the Lagrangian of the falling structure.

(b) Find the speed of the hinge when it hits the floor.
Hint: You may find the relation $\dot{\theta}\ddot{\theta} = \frac{1}{2}\frac{d\dot{\theta}^2}{dt}$ *useful for integrating the equations of motion*

(c) Find the time it takes for the hinge to hit the floor, expressing it in terms of an integral which you do not need to evaluate explicitly.



Classical Mechanics 5

Motion in One Dimension.

A particle of mass m is subject to a force $F(x, t) = F_0 t$, where F_0 is constant. The force is derivable from a potential.

(a) Find the potential energy of the particle and the Lagrangian and Hamiltonian of the particle.

(b) Write down and solve Hamilton's equations of motion subject to initial consitions $x(0) = x_0$, $\dot{x}(0) = v_0$.