Physics Comprehensive Exam - Summer 2008

Classical Mechanics

August 12, 2008
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems in Classical Mechanics. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

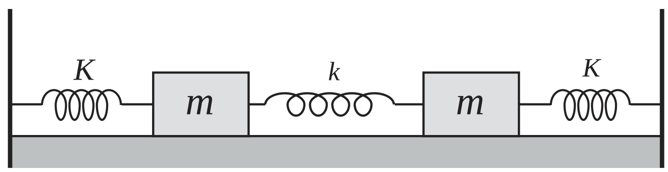
GOOD LUCK!

Name: Code Symbol:

Classical Mechanics problems: 1 2 3 4 5

Weakly Coupled Oscillators.

Two identical harmonic oscillators involve blocks of mass m attached to springs K, that are in turn attached to rigid supports. The blocks rest on a frictionless surface, and are coupled via a third spring k.



(a) Describe the normal modes of oscillation for this system. For each mode, determine both the *frequency* and the *relative displacements* of the two blocks.

(b) Assume initial conditions $x_L(0) = D$, $\dot{x}_L(0) = 0$, $x_R(0) = 0$, and $\dot{x}_R(0) = 0$. Find expressions for each block's displacement as a function of time, in terms of the normal mode frequencies and the parameter D.

(c) Assume that the coupling strength k is very weak, so that we can define a "relative strength parameter",

$$\epsilon \equiv \frac{k}{K + k} << 1$$

Let $\omega_o = \sqrt{(K + k)/m}$. (Note that this would be the natural frequency of either block, if the *other* block were held fixed.) Express the normal mode frequencies in terms of ω_o and ϵ, and show that the solution in (b) displays a low-frequency "beat phenomenon", wherein both blocks oscillate at the de-coupled frequency ω_o, but with slowly-varying "amplitude envelopes" that are 90° out of phase. Thus, the oscillation energy periodically cycles back and forth between the two blocks, at a beat frequency Ω_B:

$$x_L(t) = D \cos(\Omega_B\, t) \cos(\omega_o\, t)$$

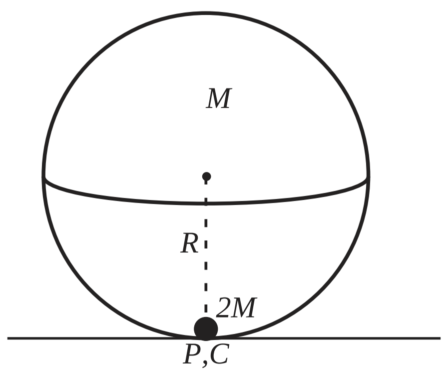
$$x_R(t) = D \sin(\Omega_B\, t) \sin(\omega_o\, t)$$

Hint: Recall that $\cos(u + v) = \cos(u)\, \cos(v) - \sin(u)\, sin(v)$.

Weebles.

A famous children's toy of the 70's and 80's was marketed with the catchy slogan that "they wobble, but they don't fall down". The toy consisted of a relatively low-mass body attached to a high-mass curved base. Provided the center of mass of the system ended up within the base, the toy would right itself even when placed on its side.

Consider the following–highly simplified–model of such a toy: a solid sphere of mass M and radius R with a point mass $2M$ permanently embedded at a fixed point on the rim. At equilibrium, the sphere rests with the point mass directly on the surface, as shown at left.

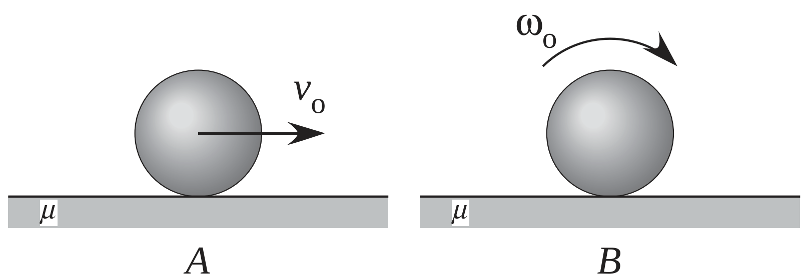


(a) Determine the Lagrangian for this system, if the sphere rocks back and forth about its equilibrium position without slipping. You may assume that the rocking occurs along a straight line (PC) on the surface, such that the equilibrium position P, the actual contact point C, the point mass $2M$, and the center of the sphere all remain within a common vertical plane (i.e., the plane of the page), as indicted at right.

(b) Find the frequency of small amplitude oscillations, as the sphere rocks back and forth.

At the Bowling Alley.

Two identical spheres of mass M and radius R are released on a flat surface having friction coefficient μ. Both spheres have an initial kinetic energy T_0. For sphere A, this initial energy is purely *translational* (along the surface): $\omega_0 = 0$, $v_0 \neq 0$, while for sphere B it is purely *rotational*: $\omega_0 \neq 0$, $v_0 = 0$.



Answer the following questions using strictly freshman-level Newtonian techniques. **Do not use Lagrangian methods!**

(a) Find the equations of motion for sphere A, and determine an expression for the time T_A required for the sphere to end up rolling without slipping.

(b) Find the equations of motion for sphere B, and determine an expression for the time T_B required for the sphere to end up rolling without slipping.

(c) Given that the two spheres started with the *same* initial kinetic energy T_0, determine:

   (i) which of the two spheres will end up rolling with the greatest translational speed;

   (ii) which of the two spheres will slide the greatest distance before it is rolling without slipping.

Classical Mechanics 4

Central Force.

A particle moves under the influence of a central force given by:

$$F(r) = -\frac{A}{r^2} - \frac{B}{r^4}$$

(a) Determine the Lagrangian for this system, and obtain the equations of motion describing the radial and angular components of the particle's motion. Are there any obvious constants of the motion?

(b) Show that the Hamiltonian for this system can be expressed in the form

$$H = \frac{p_r^2}{2m} + V_{\text{eff}}(r),$$

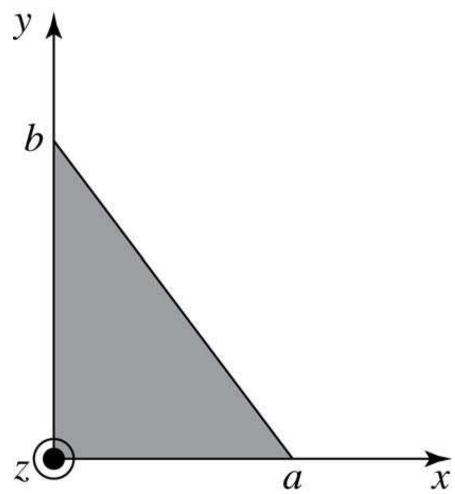
where $p_r = m\dot{r}$ is the radial component of the particle's momentum.

(c) Suppose the particle is found to be in a circular orbit of radius r_o. By considering the effective potential, find an expression for r_o in terms of the parameters A and B, and the constants of the motion. Under what conditions will this orbit be *stable* with respect to small radial deviations?

Moment of Inertia.

(a) Two circular homogeneous disks have the same mass M and the same thickness t. Disk 1 has a uniform density ρ_1, which is smaller than the uniform density ρ_2 of disk 2. Which disk, if either, has a larger moment of inertia?

(b) Consider a right triangular thin sheet with the areal density μ shown in the figure below. Find the center of mass and the components of the inertia tensor in this coordinate system.



Physics Comprehensive Exam - Winter 2007

Quantum Mechanics

August 13, 2008
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems problems in Quantum Mechanics. **DO NOT WORK ALL FIVE!**
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GOOD LUCK!

Name: Code Symbol:

Quantum Mechanics problems: 1 2 3 4 5

H-atom in Weak Electric Field.

Consider a hydrogen atom in an electric field so that the Hamiltonian becomes:

$$H = H_0 + eEr\cos\theta$$

Here the nucleus is placed at the origin and the field E is in the z-direction. The Hamiltonian for the unperturbed hydrogen atom is H_0 which is assumed to be much larger than the perturbation. Use perturbation theory with the trial wave function $\psi = \sum_i c_i f_i$, where $f_1 = |1s\rangle$ and $f_2 = |2p_z\rangle$, to determine the correction to the ground state energy for this atom. (Note: we have

$$\langle 1s|z|2p_z\rangle = \frac{a_0}{\sqrt{2}}\frac{2^8}{3^5},$$

where a_0 is the Bohr radius.)

2D Rotator.

Two masses, m_1 and m_2, which are restricted to move in a plane, are connected by a massless rod of length R to form a rigid rotator.

(a) Set up the Schrodinger equation for this rotator and obtain equations for the energy eigenvalues and the normalized eigenfunctions.

(b) Determine the eigenvalues of the operator L^2, which corresponds to the square of the angular momentum of the rotator.

Quantum Mechanics 3

Harmonic Oscillator.

At time zero a linear harmonic oscillator is in a state described by the normalized wavefunction:

$$\psi(x,0) = \frac{1}{\sqrt{5}}u_0(x) + \frac{1}{\sqrt{2}}u_2(x) + c_3u_3(x)$$

where $u_n(x)$ is the nth time-independent eigenfunction for the oscillator.

(a) Determine the numerical value of c_3 assuming it to be real and positive.

(b) Write out the wavefunction at time t.

(c) What is expectation value of the energy of the oscillator at $t = 0$? At $t = 1$?

Quantum Mechanics 4

Interference and time evolution.

Consider two quantum states specified by the following wave functions:

$$\begin{aligned}\langle x|\Psi_1\rangle &= \psi_1(x) = \left(\frac{2}{\pi a^2}\right)^{1/4} e^{-(x-a)^2/a^2}, \\ \langle x|\Psi_2\rangle &= \psi_2(x) = \left(\frac{2}{\pi a^2}\right)^{1/4} e^{-(x+a)^2/a^2}.\end{aligned}$$

Let the state of a one dimensional free particle be given at time $t = 0$ by $|\Psi(0)\rangle = N\,[|\Psi_1\rangle - |\Psi_2\rangle]$, where N is a normalization constant. Using the parity operator infer an exact expression for the probability density $|\psi(0,t)|^2$ to find the particle at $x = 0$ for all times t. Explain your reasoning.

Quantum Mechanics 5

Multi-electron atom.

Consider the possible states which may arise when more than one electron in an atom is in the same p-shell.

(a) List the possible states for 2 electrons in the same p-shell.

(b) List the possible states for 3 electrons in the same p-shell.

(c) List the possible states for 4 electrons in the same p-shell.
Hint: before you embark on something complicated for this part, think a bit!

Express your answers for the allowed states in the spectroscopic notation: $^{2S+1}L_J$, where S is the total spin of the electrons under consideration, L is the total orbital angular momentum (conventionally denoted by symbols $S, P, D, F, \cdots$ for $L = 0, 1, 2, 3, \cdots$) and J is the total angular momentum of the electrons.

Physics Comprehensive Exam - Winter 2007

Electricity & Magnetism

August 14, 2008
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems in Electricity & Magnetism. **DO NOT WORK ALL FIVE!**
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GOOD LUCK!

Name: Code Symbol:

Electricity & Magnetism problems: 1 2 3 4 5

Plasmon Dispersion Relation.

(a) Begin with the Maxwell equations in vacuum and derive the inhomogeneous wave equation satisfied by the electric field $\mathbf{E}(\mathbf{r}, t)$ in a system where $\rho(\mathbf{r}, t) = 0$ but $\mathbf{j}(\mathbf{r}, t) \neq 0$.

(b) Specialize your equation to the case of a collection of electrons (number density n) which respond independently to an electric field $\mathbf{E}$ according to Newton's law. Ignore the interactions between electrons and assume the presence of a uniform distribution of positive charge with density n, so the entire system is neutral.

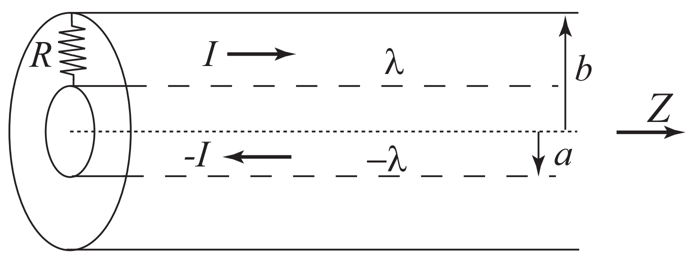
(c) Show that the equation in part (b) has a plane wave solution and derive the dispersion relation $\omega(\mathbf{k})$ explicitly.

Energy Flow in a Coaxial Cable.

A cable is made from two coaxial cylindrical shells. The outer shell has radius b, charge per unit length λ, and carries a longitudinal current I. The inner cylinder has radius $a < b$, charge per unit length $-\lambda$, and carries the current I back in the opposite direction.

(a) Integrate the Poynting vector to find the rate at which energy flows through a cross-section of the cable.

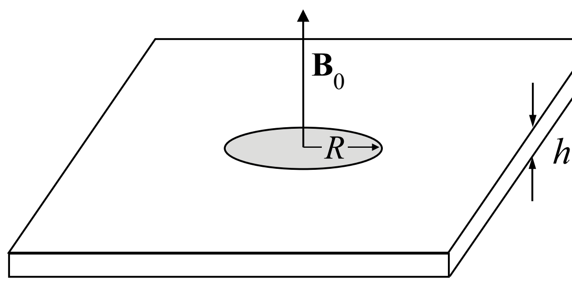
(b) Show that a resistor R connected between the cylinders dissipates the power calculated in (a).



Magnetic Film and Magnetic Disk.

(a) An infinite thin film of insulating magnetic material has permeability μ and thickness h. A uniform external magnetic field $\mathbf{B}_0$ is oriented perpendicular to the plane of the film. Find the magnetic field $\mathbf{B}$ at any point inside the film.

(b) Remove from the film in part (a) all the matter which lies outside a region of radius R (see diagram below). Find $\mathbf{B}$ at the center of the disk which remains (to first order in h/R) by subtracting, from the answer to part (a), the magnetic field produced at the center of the disk by the matter outside the disk.



Hint: The field of a point magnetic dipole with moment $\mathbf{m}$ is $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\dfrac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r} - r^2\mathbf{m}}{r^5}$.

The Electric Flux Through a Plane.

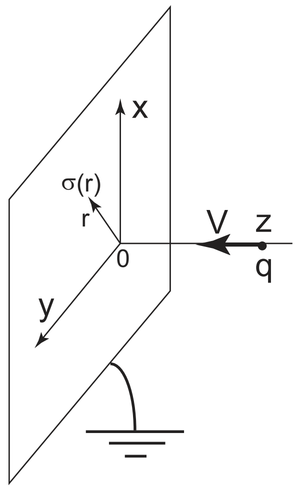
A charge distribution $\rho(\mathbf{r})$ with total charge Q occupies a finite volume V somewhere in the half-space $z < 0$. Prove that

$$\int_S dS \; \hat{\mathbf{z}} \cdot \mathbf{E} = \frac{1}{2}\frac{Q}{\epsilon_0},$$

where S is the $z = 0$ plane.

A Charge and a Plane.

A positive point charge q moves with velocity V straight towards an infinitely large, grounded, conducting plane.



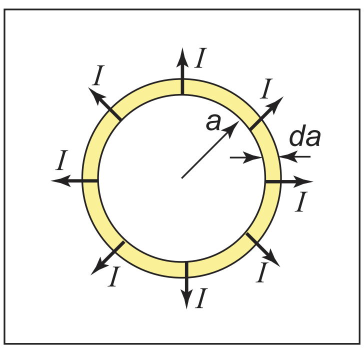
(a) Find the charge density $\sigma(r)$ on the conducting plane when the distance between the point charge and the plane is z.

(b) At what rate is Joule heat generated in the plane?

Hint: Consider a circle of radius a and estimate the rate at which the negative charge calculated in part (a) enters the circle as the charge q moves. Assume that the resistance of a ring of radius a and thickness da to a radial current is

$$R = R_S \frac{da}{2\pi a},$$

where R_S is a constant measured in ohms.



Physics Comprehensive Exam - Summer 2008

Statistical Mechanics & Thermodynamics

August 15, 2007
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems problems in Statistical Mechanics & Thermodynamics. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
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GOOD LUCK!

Name: Code Symbol:

Stat. Mech. & Thermo problems: 1 2 3 4 5

Molecular Solid.

A molecule can exist in five different states with corresponding energies $(0, 0, e, e, e)$. A solid is made of N such molecules ($N \gg 1$). The molecules are at fixed locations in space and do not interact with each other.

(a) Suppose the solid is isolated with total energy $E = ne$. Calculate the number of microstates $\Omega_N(E)$, the entropy, temperature, and heat capacity of the solid.

(b) Now suppose the solid is in contact with a reservoir at constant temperature T. Calculate the Helmholtz free energy, average total energy, and heat capacity of the solid.

(c) How does the heat capacity behave at small T? at large T? Discuss the results.

Fermion Gas.

Consider a gas of non-interacting Fermions with the energy dispersion relation $\varepsilon = ck^\alpha$, where α and c are constants, contained in a box of volume V.

(a) Calculate the grand thermodynamic potential W and the density $n = N/V$ at a constant chemical potential μ. Express your answer in terms of α and the function

$$f_m(z) = \frac{1}{(m-1)!}\int_0^\infty \frac{x^{m-1}dx}{z^{-1}e^x + 1},$$

where $z = e^{\beta\mu}$.
Hint: You may want to use integration by parts in deriving this result.

(b) Show that the ratio $PV/E = \alpha/3$.

Water Heater.

An immersion water heater of power $P = 200$ W is used to heat water in a pot. After 4 minutes, the water temperature increases from 75 °C to 85 °C. The heater is switched off for one minute and the temperature drops by 1 °C. Assuming the heat loss to the surroundings is proportional to the elapsed time, calculate the mass of water in the pot. The specific heat of water is $C = 4200$ J/kg°C.

2D Hall Conductance.

The differential free energy per unit area can be written as $dF = -B dM - S\,dT + \mu dN$, where M is the magnetic moment per unit area, and N is the number of electrons per unit area. An important observable for this system is the so-called *Hall conductance*

$$g = \frac{e}{c}\left(\frac{\partial N}{\partial B}\right)_{T,\mu} .$$

Show that an equivalent formula for this quantity is

$$g = -\frac{e}{c}\left(\frac{\partial M}{\partial \mu}\right)_{T,B} .$$

Statistical Mechanics and Thermodynamics 5

Ideal Gas.

This problem concerns an ideal gas of identical molecules of mass m at temperature T.

(a) What is the most probable speed in the Maxwell distribution? Calculate the most probable speed for an evaporated sodium gas enclosed in an oven at temperature $T = 800\,K$ (the atomic mass of the sodium atom is $m_{Na} = 23$).

(b) What is the average kinetic energy? Calculate this average kinetic energy for the sodium atoms described in question (a).

(c) How small must a gas volume be at pressure $P_0 = 10^5\,N/m^3$ and temperature $T = 300\,K$ if the root mean square deviation is to be 1% of the mean number of molecules?

Fundamental constants:

- *Boltzmann's constant:* $k_B = 1.3807 \times 10^{-23}\,J/K$,
- *nucleon mass:* $m_p = 1.67 \times 10^{-27}\,kg$,
- *electron charge:* $e = 1.602 \times 10^{-19}\,C$.