Physics Comprehensive Exam - Spring 2007

Electricity & Magnetism

March 21, 2007
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems in Electricity & Magnetism. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

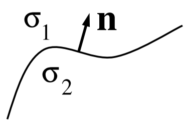
Name: Code Symbol:

Electricity & Magnetism problems: 1 2 3 4 5

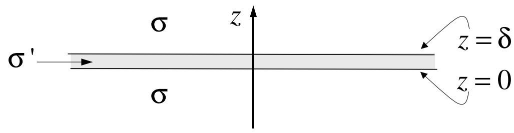
Electricity & Magnetism 1

Membrane Matching Conditions.

(a) Use the steady current condition to derive a matching condition for the electric field at the interface shown below where a unit normal $\hat{\mathbf{n}}$ separates a region with conductivity σ_1 from a region with conductivity σ_2.



(b) A thin membrane with conductivity σ' and thickness δ separates two regions with conductivity σ as shown in the figure below. Assume uniform current flow in the z-direction.



When δ is small, it makes sense to seek "across-the-membrane" matching conditions for the electrostatic potential $\varphi(z)$ defined entirely in terms of quantities defined outside the membrane. Use the results of part (a) to find the potential in all three regions of the figure and prove that the correct matching conditions are

$$\varphi(z = \delta^+) - \varphi(z = 0^-) = \delta \frac{\sigma}{\sigma'} \left. \frac{d\varphi}{dz} \right|_{z=0^-}$$

$$\left. \frac{d\varphi}{dz} \right|_{z=\delta^+} - \left. \frac{d\varphi}{dz} \right|_{z=0^-} = 0.$$

Electricity & Magnetism 2

Equal and Opposite Magnetization.

The half-space $z > 0$ has uniform magnetization $\mathbf{M} = -M\hat{\mathbf{z}}$. The half-space $z < 0$ has uniform magnetization $\mathbf{M} = +M\hat{\mathbf{z}}$. Find the magnetic field $\mathbf{B}$ at every point in space.

Electricity & Magnetism 3

A Rotating Magnet.

The equation of motion for a magnetic dipole moment **m** which rotates with an angular velocity Ω about its center is

$$\frac{d\mathbf{m}}{dt} = \Omega \times \mathbf{m}.$$

Find the electric and magnetic fields produced by this object. Neglect the displacement current in the Maxwell's equations.

Electricity & Magnetism 4

EM Waves in Low Density Plasma.

A linearly polarized, plane electromagnetic wave is normally incident on a region of space containing a low density plasma with electron density n_0. Given that the electric field is described by $\mathbf{E} = \mathbf{E_0}(\mathbf{x})e^{-i\omega t}$.

(a) Solve the equation of motion for an electron in the plasma. Assume that at such low densities, electron interactions are negligible and that $v \ll c$.

(b) Calculate the plasma's conductivity as a function of frequency.

(c) What is the index of refraction inside the plasma?

Electricity & Magnetism 5

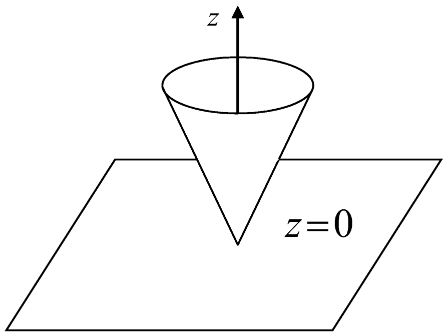
Spherical TEM Waves.

Consider time-harmonic [$\exp(-i\omega t)$] solutions to the Maxwell equations in vacuum where the fields are *independent* of the azimuthal angle ϕ. TEM solutions of this type also have no radial component to the fields: $E_r = B_r = 0$.

(a) Show that the conditions stated above decouple the Maxwell curl equations into two subsets, each of which describes a different type of TEM wave.

(b) Begin with the Maxwell divergence equations and find general solutions for $\mathbf{E}(r, \theta, t)$ and $\mathbf{B}(r, \theta, t)$ for each of the two TEM wave types.

(c) The figure below shows the apex of an infinite, solid conducting cone touching the conducting half-space $z < 0$. Explain why this structure can be used to guide one of the TEM wave types found above but not the other. Hint: You can answer this part without doing part (b).



Useful Information:

$$\nabla \cdot \mathbf{W} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 W_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta W_\theta) + \frac{1}{r\sin\theta}\frac{\partial W_\phi}{\partial\phi}.$$

$$\nabla\times\mathbf{W} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta W_\phi\right) - \frac{\partial W_\theta}{\partial\phi}\right] + \hat{\theta}\left[\frac{1}{r\sin\theta}\frac{\partial W_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rW_\phi\right)\right] + \hat{\phi}\frac{1}{r}\left[\frac{\partial}{\partial r}(rW_\theta) - \frac{\partial W_r}{\partial\theta}\right].$$

Physics Comprehensive Exam - Spring 2007

Quantum Mechanics

March 22, 2007
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems in Quantum Mechanics. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
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GOOD LUCK!

Name: Code Symbol:

Quantum Mechanics problems: 1 2 3 4 5

Quantum Mechanics 1

Perturbed Spins.

A system of two distinguishable spin 1/2 particles ($\mathbf{S}_1$ and $\mathbf{S}_2$) are in some triplet state of the total spin, with energy E_0. Find the energies of the states, as a function of λ and δ, into which the triplet state is split when the following perturbation is added to the Hamiltonian:

$$V = \lambda(S_{1x}S_{2x} + S_{1y}S_{2y}) + \delta S_{1z}S_{2z}.$$

Quantum Mechanics 2

Addition of Angular Momenta.

Various states can be created by adding angular momentum 1 to spin $\frac{1}{2}$. For this case, $|\, j_1, j_2, m_1, m_2\rangle = |\, 1, \frac{1}{2}, m_1, m_2\rangle$ where m_1 has the values +1, 0, −1 and m_2 has the values $+\frac{1}{2}$, $-\frac{1}{2}$. Below, we use the simple shorthand notation: $|\, 1, \frac{1}{2}, m_1, m_2\rangle \rightarrow |\, m_1, m_2\rangle$. Let J^2 denote the square of the total angular momentum and J_z denote the z-component of the total angular momentum (i.e. $\vec{J} = \vec{J_1} + \vec{J_2}$). Determine the J^2 and J_z eigenvalues for the two states:

$$|\, I\,\rangle \equiv \sqrt{\frac{1}{3}}\left(\left|-1, \frac{1}{2}\right\rangle + \sqrt{2}\left|0, -\frac{1}{2}\right\rangle\right)$$

$$|II\rangle \equiv \sqrt{\frac{2}{3}}\left(\left|1, -\frac{1}{2}\right\rangle - \frac{1}{\sqrt{2}}\left|0, \frac{1}{2}\right\rangle\right)$$

Hint: Recall that

$$J^2 = J_1^2 + J_2^2 + 2\, J_{1z}\, J_{2z} + J_{1+}\, J_{2-} + J_{1-}\, J_{2+}$$

$$J_{\pm}\left|\, j, m\right\rangle = \hbar\, \sqrt{(j \mp m)(j \pm m + 1)}\left|\, j, m \pm 1\right\rangle$$

Quantum Mechanics 3

Harmonic Oscillator in a Mixed State.

At $t = 0$, a harmonic oscillator state is $|\psi(0)\rangle = \frac{1}{\sqrt{3}}\,(|0\rangle + |2\rangle + |4\rangle)$.

(a) Express the state $|\psi(t)\rangle$ at arbitrary time t.

(b) Compute $\langle\psi(t)|\, x \,|\psi(t)\rangle$.

(c) Compute $\langle\psi(t)|\, x^2 \,|\psi(t)\rangle$.

Hints: $H = \hbar\omega\,(a^\dagger a + \frac{1}{2})$, $a^\dagger a\;|n\rangle = n\;|n\rangle$, and $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a)$.

Quantum Mechanics 4

δ-function Potential.

A particle of mass m moving along the x-axis is subjected to an attractive δ-function potential: $V(x) = -\alpha\,\delta(x)$. It is known that this potential admits a *single* bound state, of energy $E_o = -m\,\alpha^2/2\hbar^2$.

(a) Verify this fact, by solving the Schrödinger Equation for this system and applying the appropriate conditions on ψ and $d\psi/dx$ at the origin.

(b) Make an estimate of the lowest-energy state for this system, by choosing a (normalized) variational wave function of the form:

$$\phi(x) = \sqrt{\frac{2}{a}}\cos\left(\frac{\pi x}{a}\right) \quad \text{for} \quad -\frac{a}{2} \le x \le +\frac{a}{2},$$

where a is a parameter which may be freely varied. Is the ground state *bound*, according to your estimate?

Quantum Mechanics 5

Circle-Bound.

A particle moves in two dimensions in a circularly-symmetric potential. The wave function for the particle is

$$\psi(r, \theta) = N\, e^{-\lambda r}\left(1 + e^{2i\theta}\right)$$

where λ is a constant and N is the normalization.

(a) What is the probability that a measurement of the angular momentum L_z would yield a value 0?

(b) What is the mean value of L_z?

(c) If $\vec{L} = \vec{r} \times M\vec{v}$, where $\vec{v} = (v_r, v_\theta)$ in cylindrical coordinates, find the mean orbital speed $\langle |v_\theta| \rangle$.