# Physics Comprehensive Exam - Fall 2007

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## Classical Mechanics

August 14, 2007
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems in Classical Mechanics. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: Code Symbol:

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Classical Mechanics problems: 1 2 3 4 5

**Bead on a Wire.**

A bead of mass $m$ slides without friction on a wire that is bent in the shape of a parabola $z = k\rho^2$ in cylindrical polar coordinates. The wire is being spun with constant angular velocity $\omega$ about its vertical axis of symmetry. Write down the Lagrangian in terms of generalized coordinate $\rho$. Find the equation of motion of the bead and determine whether there are positions of equilibrium, that is, values of $\rho$ at which the bead can remain fixed, without sliding up or down the spinning wire. Determine for different $\omega$ the stability of all equilibrium positions.

**Particle in a Spherically Symmetric Potential.**

A particle moves in the potential,

$$V(r) = Kr^4,$$

where $K$ is a constant, with angular momentum $L$ and energy $E$.

(a) Determine the radius $r_0$ of circular orbits in terms of $L$.

(b) Determine the energy $E_0$ of circular orbits in terms of $r_0$.

(c) Calculate the period $T_\theta$ for circular motion in terms of $r_0$.

**Rotating Disc.**

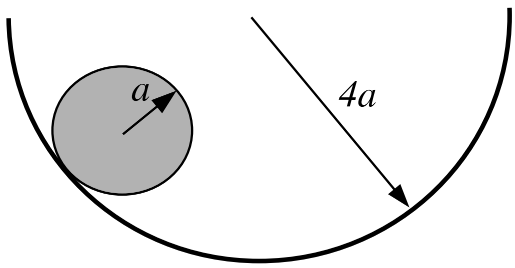
A thin uniform disc of radius $a$ and mass $m$ is rotating freely on a frictionless bearing with uniform angular velocity $\omega$ about a fixed vertical axis passing through its center, and inclined at angle $\alpha$ to the symmetry axis of the disc. Neglecting gravity, determine the magnitude and direction of the torque required to keep the disc rotating at a constant angular velocity.

**Mass on a Spring.**

A particle of mass $m$ lies on a smooth horizontal table and is attached by a linear spring to a fixed point on the table. The particle is fired with speed $v_0$ perperdicular to the unstretched spring along the table. If the spring has modulus of elasticity $k = 3m^2v_0^2/4a^2$ where $a$ is the natural length of the spring, find the radii of the circles–the annulus-within which the particle moves.

**Rolling Cylinder.**

A uniform circular cylinder of radius $a$ and mass $m$ rolls without slipping on the inner surface of a fixed cylinder of radius $4a$. Find the period of small oscillations of the rolling cylinder.



# Physics Comprehensive Exam - Fall 2007

## Electricity & Magnetism

August 15, 2007
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems in Electricity & Magnetism. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
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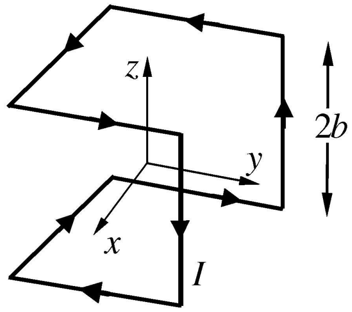
GOOD LUCK!

Name: Code Symbol:

Electricity & Magnetism problems: 1 2 3 4 5

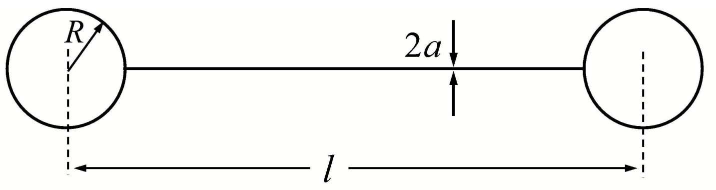
**Purcell's Loop.**

A filamentary current loop traverses eight edges of a cube with side length $2b$ as shown below. Find the magnetic dipole moment $\mathbf{m}$ of this structure. *Hint: Use superposition and the fact that a planar loop of area $A$ which carries a current $I$ produces a magnetic moment with magnitude $IA$.*



**A Dumbell-Shaped Oscillator.**

An electromagnetic oscillator is formed when charge flows back and forth between two identical perfectly conducting spheres of radius $R$ connected by a very thin and very long perfectly conducting rod of radius $a \ll R$ and length $l \gg R$. At one moment of the oscillation cycle, one sphere has charge $+Q$, one sphere has charge $-Q$, and no current flows through the rod.




(a) Estimate the capacitance of this system.

(b) Estimate the inductance of this system. Hint: Use an energy method.

(c) Show that the resonant frequency of the oscillator is

$$\omega \approx \frac{c}{\sqrt{Rl \ln(l/a)}}.$$

# *Electricity & Magnetism 3*

**Electromagnetic Duality.**

(a) Let $\mathbf{E}$ and $\mathbf{B}$ solve the free space Maxwell equations. Define a *dual* magnetic field as $\mathbf{B}' = \mathbf{E}/c$. Find a dual electric field $\mathbf{E}'$ so that $\mathbf{E}'$ and $\mathbf{B}'$ solve the free-space Maxwell equations.

(b) How are the electromagnetic momentum density and Poynting vector of the two solutions in part (a) related to one another?

(c) How are the primed and unprimed solutions related for the case of propagating plane waves?

**Electric and Magnetic Mass.**

A tiny thin spherical shell with radius $a$ and uniformly distributed charge $q$ moves with constant velocity $\mathbf{v}$. For any $\mathbf{v}$, the electric and magnetic fields produced by the shell are related by $\mathbf{B} = (1/c^2)\mathbf{v} \times \mathbf{E}$. Work in the quasi-electrostatic approximation where $v \ll c$ and let $U_E$ and $U_B$ be the total electric and magnetic energies of the shell. Define a “rest mass” $m_0$ of the shell from $U_E = m_0 c^2$. Define a “kinetic mass” $m$ from $U_B = \frac{1}{2} m v^2$.

(a) Show that $m > m_0$.

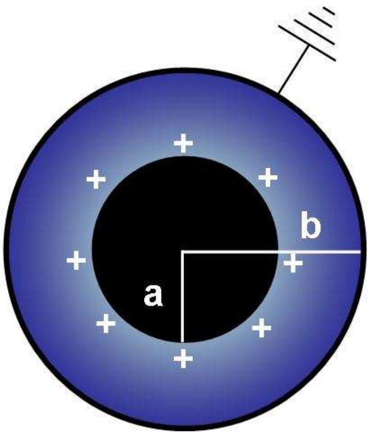
(b) Describe the flow of energy for this situation predicted by the Poynting vector.

**Spherical Capacitor.**

The volume between two concentric conducting spherical surfaces of radii $a$ and $b$ ($a < b$) is contains a material with a radially-dependent dielectric constant:

$$\varepsilon = \frac{\varepsilon_0}{1 + Kr} \ ,$$

where $\varepsilon_0$ and $K$ are constants and $r$ is the radial coordinate.




A charge $Q$ is placed on the inner surface, while the outer surface is grounded. Given this information, find the following quantities:

(a) The electric displacement field in the region $a < r < b$;

(b) The capacitance of the device;

(c) The polarization charge density in $a < r < b$;

(d) The surface polarization charge density at $r = a$ and $r = b$.

# Physics Comprehensive Exam - Fall 2007

## Quantum Mechanics

August 16, 2007
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems problems in Quantum Mechanics. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
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5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: Code Symbol:

Quantum Mechanics problems: 1 2 3 4 5

**Coupled Harmonic Oscillators.**

Consider a coupled pair of one-dimensional, distinguishable, simple harmonic oscillators with equal masses, equal individual potentials $U(x_1) = \frac{1}{2}Cx_1^2$, $U(x_2) = \frac{1}{2}Cx_2^2$, $C > 0$, and a coupling potential $U_c(x_1, x_2) = \frac{1}{2}k(x_1 - x_2)^2$, $k > 0$. The Hamiltonian for this system is:

$$H = T + U = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}C(x_1^2 + x_2^2) + \frac{1}{2}k(x_1 - x_2)^2$$

(a) Show that by a suitable change of variables, $H$ takes the form:

$$H = \frac{P^2}{2M} + \frac{A}{2}R^2 + \frac{p^2}{2\mu} + \frac{B}{2}r^2.$$

*Hint: you may find the identity* $u^2 + v^2 = \frac{1}{2}[(u+v)^2 + (u-v)^2]$ *to be of use, here.*

(b) Show that the total eigenfunction can be written as a product of two functions and determine the energy eigenvalues of the coupled system of distinguishable particles.

(c) Now assume that the particles are indistinguishable so that they are either Bosons or Fermions. Using the symmetry properties of the product functions determined in part (b), determine which of the energy levels found in part (b) are associated with each type of particle.

(d) Show that for Fermions with their spins aligned the probability of finding the two particles at the same position is zero.

(Note: In this problem, you may use your knowledge of the wavefunctions and eigenvalues of the simple harmonic oscillator without derivation or proof.)

# *Quantum Mechanics 2*

**Measuring Spin.**

A spin-1/2 particle is prepared in a state $\chi_o$, such that it's spin is quantized along an axis defined by:

$$\hat{n} = +\frac{2}{3}\,\hat{\imath} + \frac{2}{3}\,\hat{\jmath} - \frac{1}{3}\,\hat{k}$$

(a) Express the spin component $S_{\hat{n}} \equiv \vec{S} \cdot \hat{n}$ as an operator, using the Pauli matrices as a basis. Express the state $\chi_o$ in terms of the standard $z$-quantized spinors ($\chi_+$ and $\chi_-$).

(b) Find the probabilities of measuring the spin to be aligned along: (i) the $+\hat{\imath}$ axis; (ii) the $+\hat{\jmath}$ axis; and (iii) the $+\hat{k}$ axis.

Recall that the Pauli matrices are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$$

# *Quantum Mechanics 3*

**3D Harmonic Oscillator.**

The isotropic, three-dimensional harmonic oscillator can be characterized using one of two bases: (i) a direct product of three separate 1D oscillator bases (that is, $|N\rangle = |n_x\rangle|n_y\rangle|n_z\rangle$ ), having a total energy $E = (n_x + n_y + n_z + 3/2)\,\hbar\omega = (N + 3/2)\,\hbar\omega$; or (ii) a basis in which the total energy **and** the orbital angular momentum $|N, \ell m\rangle$ are well defined. (This latter basis is a consequence of the spherical symmetry of the Hamiltonian).

(a) Find an expression for the parity of a 1D oscillator state $|n\rangle$.

(b) Using you answer from (a), identify the parity and degeneracy of each of the three lowest energy levels for the 3D oscillator.

(c) Use your knowledge of the degeneracy and parity of the angular momentum states $(\ell m)$ to deduce the allowed angular momentum values for each of the three lowest energy levels of the 3D oscillator.

**(Almost) Cubic Potential Well.**

Consider a particle of mass $m$, bound within a three-dimensional, infinitely deep cubical well, defined by:

$$V(x, y, z) = \begin{cases} 0, & \text{if } 0 < x < L,\ 0 < y < L,\ 0 < z < L \\ \infty, & \text{otherwise} \end{cases}$$

(a) What are the energy eigenstates for this system? Identify the four lowest-lying states, and their energies.

(b) Suppose that we perturb the well by putting a $\delta$-function "bump" at the position $x = \frac{3L}{4}$, $y = \frac{L}{2}$, $z = \frac{L}{4}$:

$$V' = A\,\delta\left(x - \frac{3L}{4}\right)\delta\left(y - \frac{L}{2}\right)\delta\left(z - \frac{L}{4}\right)$$

Find the first-order corrections to the energies of the four lowest-lying states.

**Electron in a Uniform Magnetic Field.**

Consider an electron at rest in the presence of a uniform magnetic field $\vec{\mathbf{B}} = B_z\hat{\mathbf{z}}$ for which the Hamiltonian is:

$$H_0 = \frac{eB_z}{m} S_z$$

Now turn on a perturbation in the form of a uniform field along the $x$-direction:

$$H' = \frac{eB_x}{m} S_x$$

(a) Find the matrix elements of $H'$ in the basis of eigenstates of $H_0$.

(b) Calculate corrections to the ground state energy to first non-vanishing order in perturbation theory.

(c) Find the **exact** ground state energy of $H_0 + H'$.

(d) Compare your answers from parts (b) and (c). Do they agree? If yes, explain why. If not, explain why not.

# Physics Comprehensive Exam - Fall 2007

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## Statistical Mechanics & Thermodynamics

August 17, 2007
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems problems in Statistical Mechanics & Thermodynamics. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
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5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: Code Symbol:

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Stat. Mech. & Thermo problems: 1 2 3 4 5

**Mixing Gases.**

Two vessels $A$ and $B$ each contain $N$ molecules of the same perfect monatomic gas. Initially the two vessels are thermally isolated from each other, the two gases being at the same pressure $P$, and at temperatures $T_A$ and $T_B$, respectively. The two vessels are now brought into thermal contact, the pressure of the gases being kept constant at the value $P$. Find the change in the entropy of the system after equilibrium is established, and show that this change is non-negative.

**Chain Polymer.**

In a simple quantized model of rubber elasticity, a polymer chain of $N$ links, each of length $a$, is subject to a force $F$ in the $+x$ direction. Each link can point independently along any one of the $x$, $y$ and $z$ axes, either in the (+) or (−) direction. The energy is only $x$-dependent; $\varepsilon = aF$ for the link pointing along $x$, $\varepsilon = -aF$ for the link along $+x$, and $\varepsilon = 0$ for the link along $y$ and $z$.

(a) Calculate the partition function for the $N$-link chain.

(b) Calculate the expectation value for the chain length.

(c) Show that the linear coefficient of thermal expansion is negative, as in the case of rubber.

**Photon Gas.**

Consider a gas of photons contained inside a box of volume $V$ in equilibrium at temperature $T$. Calculate its

(a) pressure $P$,

(b) entropy $S$, and

(c) chemical potential $\mu$.

Hint: The energy of a photon is $E = c\hbar k$.

**One-dimensional Crystal.**

The vibrational spectrum of a one-dimensional crystal is given by

$$\omega_n = \Omega\,[2(1 - \cos(2\pi n/N))]^{1/2}\,,$$

where $n$ is an integer ranging from $-N/2$ to $N/2$.

(a) Calculate the specific heat $c_V$ at large $T$.

(b) For $T \to 0$, the heat capacity varies as $c_V \propto \Omega^{-\alpha} T^{\gamma}$. Compute $\alpha$ and $\gamma$.

**Vapor Pressure of a Liquid Droplet.**

For a small droplet the ratio of area to volume increases greatly, and surface phenomena become important. Consider a liquid of molecular mass $M$, density $\rho$, and surface tension (surface energy per unit area) $\gamma$. A large volume of this liquid (i.e., having negligible surface-to-volume ratio) has equilibrium varpor pressure $P_\infty$ at temperature $T$. At the same temperature, a droplet of diameter $d$ has vapor pressure $P_d$. Assuming that the difference between $P_d$ and $P_\infty$ is due to surface tension alone, use the Gibbs-Duhem relation

$$N d\mu + S\,dT - V dp = 0.$$

to determine $P_d$.

You may assume that the density of the vapor is much smaller than that of the liquid and that the vapor is a perfect gas. Further, you may assume that $|P_d - P_\infty| \ll \gamma/d$.