Physics Comprehensive Exam - Fall 2006

Classical Mechanics

August 15, 2006
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems in Classical Mechanics. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
4. Start each new problem at the top of a fresh sheet of paper.
5. Explain your reasoning whenever possible.

GOOD LUCK!

Name: Code Symbol:

Classical Mechanics problems: 1 2 3 4 5

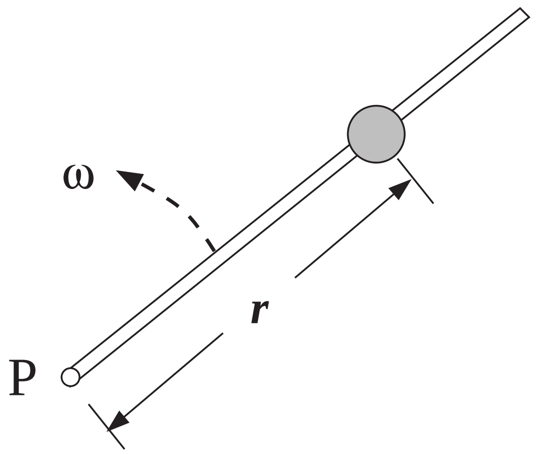
Motion of a bead.

A bead of mass m slides without friction on a rod that is made to rotate at a constant angular velocity ω in the plane of the paper (see figure below). Neglect gravity.

(a) Write the Lagrangian of the system.

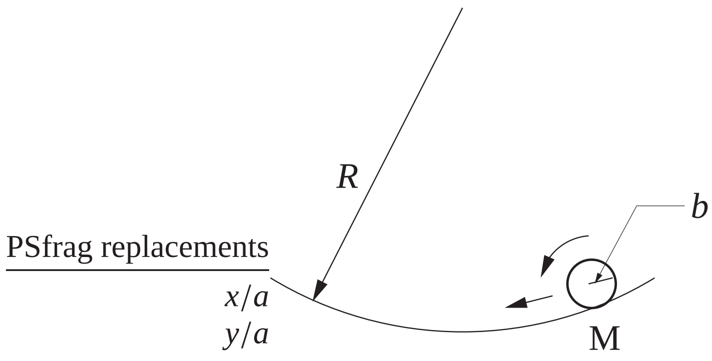
(b) Show that $r = r_0 \exp(\omega t)$ is a possible motion of the bead, where r_0 is the initial distance of the bead from the pivot point P.

(c) For the motion described in part (b), find the generalized forces exerted on the bead by the rod.



Classical Mechanics 2

Motion of a marble. A marble (ball) of radius b rolls back and forth in a shallow dish of radius R. Find the frequency of small oscillations, assuming that $R \gg b$. (See figure below).



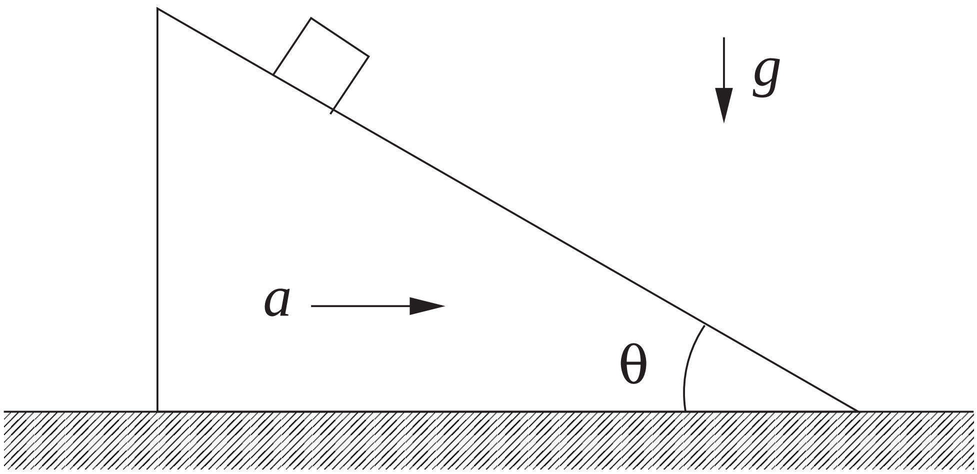
Sliding Block.

A block rests on a wedge inclined at an angle θ as shown in the figure below. The coefficient of friction between the block and the plane is μ.

(a) Find the maximum value of θ for the block to remain motionless if the wedge is fixed in position, i.e. the wedge does not move.

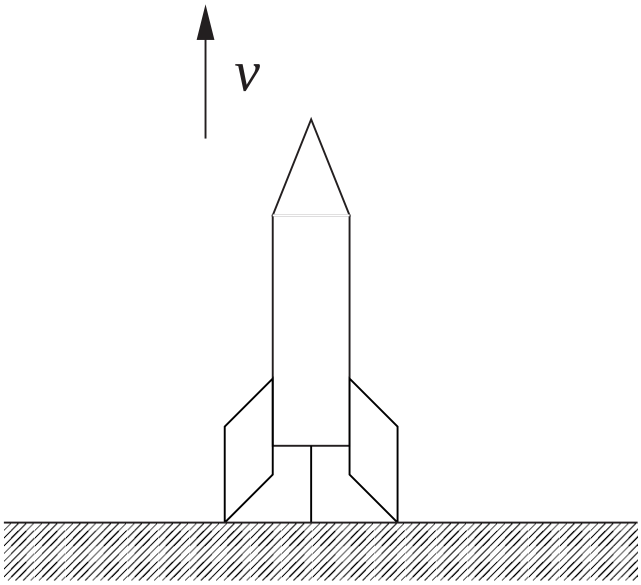
(b) Let the wedge move with horizontal acceleration a, as shown in the figure below. Assuming that $\tan\theta < \mu$, find the minimum value of a for the block to remain on the wedge without sliding.

(c) Repeat part b, but find the maximum value of the acceleration.



Rocket Motion.

A rocket ascends from rest in a uniform gravitational field by ejecting exhaust with constant speed u. Assume that the rate at which mass is expelled is given by $dm/dt = \gamma m$, where m is the instantaneous mass of the rocket, γ is a constant. In addition, the rocket is retarded by air resistance with a force mbv, where b is a constant. Find the velocity v of the rocket as a function of time.



Charged Harmonic Oscillator.

The Hamiltonian of a particle with charge q in a uniform magnetic field $\mathbf{B}$ along the z-direction is

$$H = \frac{1}{2m}(p + \frac{q}{c}\mathbf{A})^2,$$

in Gaussian units, where

$$\mathbf{A} = \frac{1}{2}B(-y, x, 0).$$

(a) Construct the Lagrangian $L(x, y, \dot{x}, \dot{y})$.

(b) Obtain the equations of motion for x and y.

(c) Solve the coupled equations for the normal mode frequencies.

Physics Comprehensive Exam - Fall 2006

Electricity & Magnetism

August 16, 2006
9:00 am – 2:00 pm

Instructions:

1. Work **four** of the five problems in Electricity & Magnetism. **DO NOT WORK ALL FIVE!**
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GOOD LUCK!

Name: Code Symbol:

Electricity & Magnetism problems: 1 2 3 4 5

How to Measure E.

Students are often told that $\mathbf{E} = \mathbf{F}_q/q$ defines the electric field at a point if $\mathbf{F}_q$ is the measured force on a tiny charge q placed at that point. Careful instructors let $q \to 0$ to avoid the polarization of nearby matter due to the presence of q. Unfortunately, this experiment is impossible to perform. A better definition uses $\mathbf{F}_q$ and the force $\mathbf{F}_{-q}$ measured when $-q$ sits at the same point. There is no need to let $q \to 0$, even if conductors or (linear) dielectric matter is present. Derive an expression that relates $\mathbf{E}$ to $\mathbf{F}_q$ and $\mathbf{F}_{-q}$.

Plane Wave?.

At the very least, any valid time-dependent electric field in vacuum must solve

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.$$

Prove that $\mathbf{E}(x, y, z, t) = \hat{\mathbf{x}} E_0(x, y) e^{i(kz-\omega t)}$ with $\omega = ck$ is a solution to this equation if and only if $E_0(x, y) = const$.

What other conditions, if any, must $\mathbf{E}$ satisfy if it is to be a legitimate electric field?

A Localized Current.

A current distribution is confined to a region of space near the origin with linear size d. When $r \gg d$, the magnetic field is

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{\mathbf{a} \times \mathbf{r}}{r^3} (1 - ikr) e^{i(kr - \omega t)}.$$

The vector **a** is constant in space and time.

(a) Focus on the part of space where $kr \ll 1$. Explain why the Biot-Savart law can be used in that regime to find the magnetic field. Use it to relate **a** to the current density $\mathbf{j}(\mathbf{r}, t)$ that generates the magnetic field.

(b) Focus on the part of space where $kr \gg 1$. Find the radiation electric field.

Hint: $\nabla \times (f\mathbf{V}) = f(\nabla \times \mathbf{V}) - \mathbf{V} \times \nabla f$

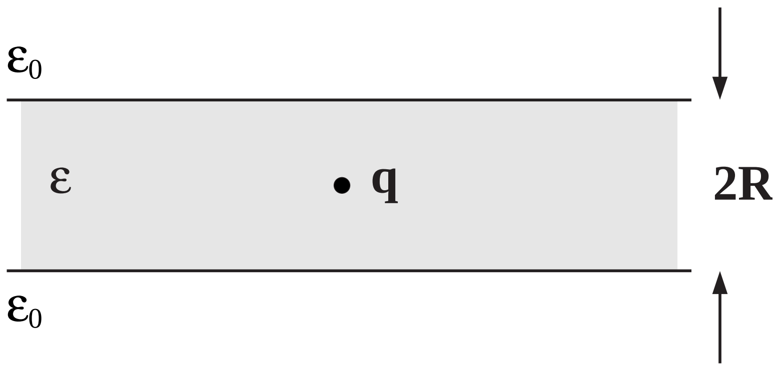
Dielectric Tube.

A point charge q is placed on the axis of an infinitely long circular cylinder of radius R made of a dielectric material with very high permittivity $\epsilon/\epsilon_0 \to \infty$.

(a) Sketch the field lines of the electric field everywhere.

(b) Find an approximate expression for **E** inside the tube at a distance from the point charge that is very much larger than R.

Hint: If you cannot do part (b), find the electric flux through a cross section of the tube at a distance from the point charge that is very much larger than R.

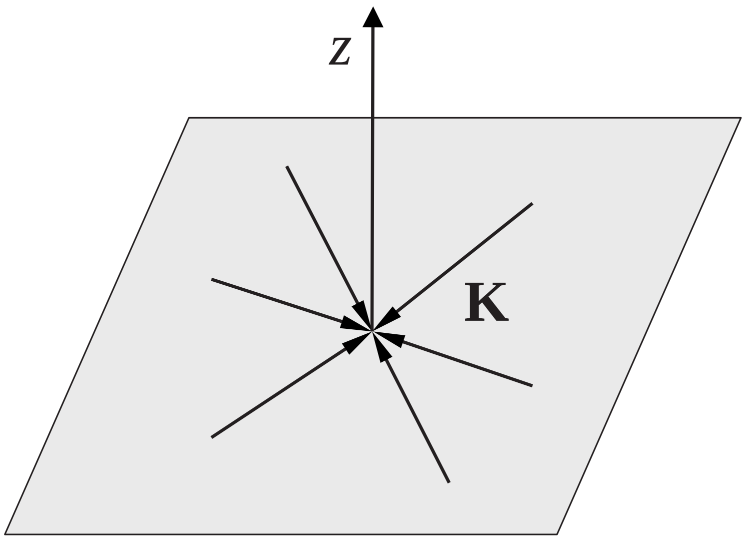


Displacement Current.

A time-independent surface current with density **K** flows in the x-y plane from infinity to the point $\mathbf{r} = 0$ in a radially symmetric manner. As a result, charge accumulates at $\mathbf{r} = 0$ at the rate $dq/dt = I$.

(a) Find the displacement current.

(b) Find the total magnetic field everywhere.



Physics Comprehensive Exam - Fall 2006

Quantum Mechanics

August 17, 2006
9:00 am – 2:00 pm

Instructions:

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Quantum Mechanics problems: 1 2 3 4 5

Atomic Internal Conversion.

Certain nuclei can occasionally de-excite by internal conversion, which is a process whereby the excitation energy is transferred directly to one of the atomic electrons, causing it to be ejected from the atom. It is a reasonable assumption that the probability of this occurrence on a particular electron is directly proportional to the probability of that electron being at the nucleus.

(a) Of the $n = 2$ electrons, which are the most likely to undergo internal conversion and why?

(b) Estimate the ratio of the conversion rate for 1s electrons to the conversion rate for 2s electrons.

The radial part of the electronic wavefunctions for $n = 1, 2$ is given by

$$R_{10}(r) = 2\frac{Z}{a_0}e^{-\frac{Zr}{a_0}}, \qquad R_{20}(r) = \left(\frac{Z}{2a_0}\right)^{\frac{3}{2}}\left(2 - \frac{Zr}{a_0}\right)e^{-\frac{Zr}{2a_0}}, \qquad R_{21}(r) = \frac{1}{\sqrt{3}}\left(\frac{Z}{2a_0}\right)^{\frac{3}{2}}\frac{Zr}{a_0}e^{-\frac{Zr}{2a_0}}.$$

Quantum Mechanics 2

Variational Wavefunction.

A particle of mass M is subjected to a potential of the form $V = \infty$ for $x < 0$ and $V = bx$ for $x > 0$. Let the trial wavefunction be $\phi(x) = x \exp(-cx)$, where c is a variational parameter. Using this function, find the best estimate of the ground state energy.

Charged particle on a ring.

Consider a charged particle on a ring of unit radius. A magnetic field is arranged so a magnetic field flux $\phi = \alpha\phi_0$ passes through the ring, where $\phi_0 = hc/e$ and α is a real number. Here, h is Planck's constant, c is the speed of light, and e is the electron charge.

The Hamiltonian operator in units of $\hbar = 2m = 1$ can be written as $H = H_0 + V$, where

$$H_0 = \left(i\frac{\partial}{\partial\theta} + \alpha\right)^2$$

and $V = V_0 \cos\theta$, with θ being the angular coordinate.

(a) Find the complete set of eigenvalues and eigenfunction of H_0.

(b) Use perturbation theory to find the first and second order corrections to the ground state energy E_0 due to the perturbation V for $0 < \alpha < 1/2$.

(c) For $\alpha = 1/2$ the ground state of H_0 is degenerate. Find the first order correction to E_0 duer to the perturbation V.

Quantum system with two states.

Consider a two-state quantum system with energy eigenvalues $\hbar\omega_a$ and $\hbar\omega_b$ corresponding to (properly normalized) eigenstates $|a\rangle$ and $|b\rangle$.

(a) Consider a mixed state, such that at $t = 0$ the system is in the state $\psi_1 = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$ with 25% probability, in the state $\psi_2 = \frac{1}{\sqrt{2}}(|a\rangle - |b\rangle)$ with 25% probability, and in the state $\psi_3 = \frac{1}{\sqrt{2}}(|a\rangle + i|b\rangle)$ with 50% probability.

The **density operator** for a statistical mixture of states is defined by $\rho = \sum_k p_k \rho^k$, where p_k is the probability of being in state $|\psi_k\rangle$ and $\rho^k = |\psi_k\rangle\langle\psi_k|$ is the pure state density operator. Find the matrix elements of ρ in the basis of $|a\rangle$ and $|b\rangle$ at $t = 0$.

(b) Explain how the density operator evolves in time in the Schrödinger and in the Heisenberg representation. Find the density matrix (matrix representation of ρ defined above) at time t for the mixed state in part (a) in each representation.

Identical Fermions.

The Hamiltonian for two interacting identical fermions (spin 1/2) in one dimension is

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{m\omega^2}{2}(x_2 - x_1)^2.$$

(a) What is the energy spectrum?

(b) What are the corresponding eigenfunctions (including the spin state)?

Physics Comprehensive Exam - Fall 2006

Statistical Mechanics & Thermodynamics

August 18, 2006
9:00 am – 2:00 pm

Instructions:

1. Work **four** problems in Statistical Mechanics & Thermodynamics. **DO NOT WORK ALL FIVE!**
2. On this cover sheet, circle the problems that you have chosen to hand in.
3. Clearly write your code symbol - NOT YOUR NAME - at the top of each sheet of paper you hand in.
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GOOD LUCK!

Name: Code Symbol:

Stat. Mech. & Thermo problems: 1 2 3 4 5

Bose Gas.

Assume that the ground state energy is **zero** for a Bose ^{4}He gas with N particles. When T→0 the chemical potential $\mu(T = 0)$ is much less than the first excited state energy (almost all atoms are in the ground state).

(a) Show that as T→0

$$e^{\mu/kT} \approx 1 - \frac{1}{N}. \tag{1}$$

(b) The total number of particles is

$$N = \sum D(E)\langle n(E)\rangle$$

where $\langle n(E)\rangle$ is the statistical average of the particle number and where the density of states, $D(E) = 1$, when $E = 0$; and $D(E) = \alpha E^{1/2}$, when $E > 0$. Here $\alpha = (V/4\pi^2) \times (2m/\hbar^2)^{3/2}$.

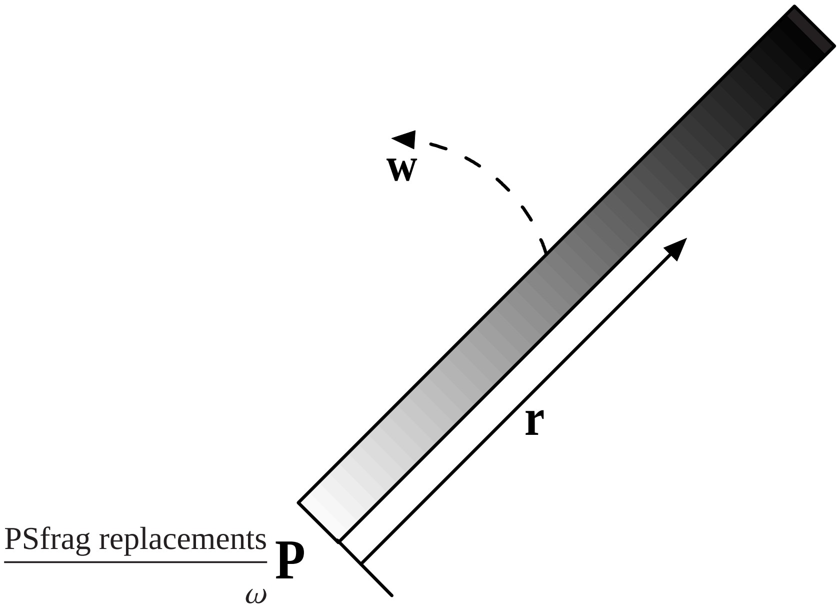
Let N_0 be the number of atoms in the ground state and N_e be the number of atoms in excited states so $N = N_0 + N_e$. Calculate $N_e(T)$ assuming that T is low enough for Eq. 1 to hold and that $N \gg 1$. Write your answer in terms of a definite integral.

(c) If the condensation temperature, T_e, is defined as the temperature above which most atoms are in excited states (i.e. $N_e(T_e) \approx N$), write an expression for T_e assuming $T_e \approx 0\ K$.

DNA Solution.

A solution of DNA molecules at temperature T and unknown molecular weight is placed in a tube in a centrifuge. The centrifuge rotates about the pivot point P at a fixed angular frequency ω (See figure below).. Because of the centripetal acceleration, the DNA density, $\rho(r)$ in the tube at equilibrium is a function of distance r from the rotation axis.

Show that by measuring the density at two points, r_1 and r_2, the molecular weight defined as $\mu = N_a m$ (m is the mass per molecule and N_A is Avogadro's number) is uniquely determined.



Nernst versus Planck.

The Third law of thermodynamics can be stated in 2 ways;

$$\lim_{T\to 0}\left(\frac{\partial G}{\partial T}\right)_P = \lim_{T\to 0}\left(\frac{\partial H}{\partial T}\right)_P \text{ and } \lim_{T\to 0} G = \lim_{T\to 0} H \quad \text{The Nernst hypothesis}$$

$$\lim_{T\to 0} S = 0 \quad \text{The Planck Statement}$$

where G is the Gibb's function and H is the enthalpy.

(a) Define a new thermodynamic variable $\Phi = G - H$. Using the Nernst hypothesis, find Φ and $(\partial\Phi/\partial T)_P$ as T goes to zero.

(b) Using this fact, show that Nernst's hypothesis leads to Planck's statement of the third law.

HINT:L'Hôpital's rule states that the limit of $\lim_{T\to 0}\Phi/T$ is the same as $(\partial\Phi/\partial T)_P$ when $T \to 0$. Therefor, consider the function $(\partial\Phi/\partial T)_P - \Phi/T$.

Statistical Mechanics 4

Spins in a Magnetic Field.

Consider a system of N distinguishable non-interacting spins in a magnetic field B. Each spin has a magnetic moment of size μ, and each can point either parallel or antiparallel to the field. Thus, the energy of any particular configuration of spins is:

$$E_i = -n_i \mu B, \quad n_i = \pm 1, \quad i = 1, ..., N$$

(a) Derive an expression for the average energy $\langle E \rangle$ for this system.

(b) Derive an expression for the entropy S for this system.

(c) Suppose that a sample of this material is placed in contact with a thermal reservoir at $T_0 = 4\,\text{K}$, and is subjected to a magnetic field of magnitude $B_0 = 1.0\,\text{T}$. After equilibrium is reached, the sample is thermally isolated, and then the magnetic field is reduced to $B_1 = 1.0\,\text{mT}$. What happens to the sample?

Equation of State of an Exotic Gas.

A hypothetical gas has an isothermal compressibility $\kappa = a/(2v^2)$ and an isobaric expansivity $\beta = (3bT^2)/(2v^2)$, where a and b are empirical constants and v is the molar volume.

Find an equation of state for this gas, in the form:

$$f(v, P, T) = \text{constant}.$$

HINT: Recall that the compressibility and expansivity are defined by the relations:

$$\kappa \equiv -\frac{1}{v}\Big(\frac{dv}{dP}\Big)_T \quad \text{and} \quad \beta \equiv \frac{1}{v}\Big(\frac{dv}{dT}\Big)_P.$$