Homework Assignment #7

1. Inertial frame of reference is defined as a frame in which space is homogenous and isotropic and time is homogenous. Show that a free motion takes place at constant velocity in an inertial frame.
2. Show that Lorentz transformations can be understood as rotations in time-space plane of Minkowsky 4-dimensional space.
3. Using the previous result derive Lorentz transformations for energy and momentum.

1. A body is said to be in free motion if there are no external forces acting on it. We wish to show that in an inertial reference frame, i.e. a frame in which space is homogeneous and isotropic and time is homogeneous, free motion takes place with constant velocity. The line of reasoning is based on the following definitions:

   Homogeneous space – There is no special point in space. All points are equally valid. That is, an observer in an inertial frame cannot find any difference between the properties of any two points in space.

   Homogeneous time – There is no special point in time. All instants of time are equally valid. That is, all instants are equally important to the observers.

   Isotropic space – There is no special direction in space.

   The fact that free motion in such a frame takes place with constant velocity can be shown with counter-examples:

   - Suppose an object 'A' in free motion does not have constant velocity. This means that A has a non-zero acceleration. Suppose that at time $t = t_0$ the velocity of A is $v = 0$. Then, because A has non-zero acceleration, its velocity at time $t = t_0 + a$, where a is some small number, is non-zero. Thus, two different instants of time yield a different velocity, even though the object is in free motion (i.e., no external forces are acting on it). This violates the principle of homogeneity of time.

   - We again suppose that A is in free motion and does not have constant velocity. This means acceleration is non-zero. Then, consider the following relationships:

     $$a = \frac{dv}{dt} = \frac{dv}{dx} * \frac{dx}{dt} \neq 0 \quad \Rightarrow \quad \frac{dv}{dx} \neq 0$$

     This argument shows that $\frac{dv}{dx}$ is non-zero, i.e., velocity changes with position. This implies that with no forces acting on it, A is somehow able to distinguish between points in space, because it can observe a change in its velocity as position changes. Thus, all points are somehow not equivalent. This violates the principle of homogeneous space.

   - Finally, suppose there is a special direction in space. Then, suppose A is suddenly immersed in this space. Whatever A's initial orientation, it will rotate and align itself with the special direction, even though no forces are acting on it. Changing directions implies a non-constant velocity. If there are no special directions, an object will keep its initial orientation, and thus have constant velocity. This shows that isotropic space yields constant direction, which is an important component of constant velocity.

2. First, let us consider Euclidean space. Consider two coordinate systems, K and K', that are rotated with respected to each other by an angle θ. Consider two points P_1 and P_2, with coordinates (x_1,y_1) and (x_2,y_2) in frame K, and (x_1',y_1') and (x_2',y_2') in frame K'. The following figure illustrates this:

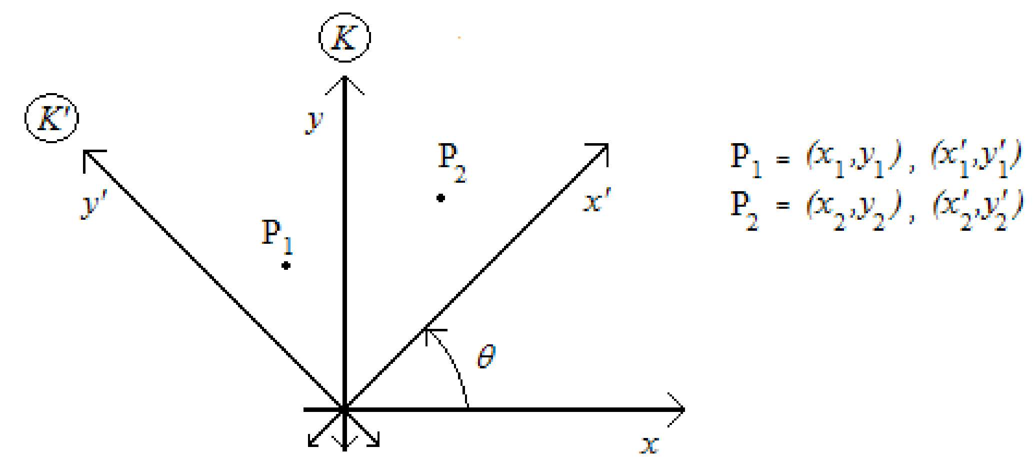

Figure 1. Points P_1 and P_2 in frames K and K'.

The distance L between P_1 and P_2 in frame K is: $L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The distance L' between P_1 and P_2 in frame K' is: $L = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}$

The characteristic property of Euclidean space is that $L = L'$.

It can be shown that the following transformation for rotations preserves this property:

$$x' = x\cos\theta + y\sin\theta$$
$$y' = y\cos\theta - x\sin\theta \qquad \text{Eqn (1)}$$

In Minkowsky 4-dimensional space, the preserved quantity analogous to Euclidean distance is the space-time interval:

$$s^2 = c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2, \qquad \text{Eqn (2)}$$

where (x_1,y_1,z_1,t_1) and (x_2,y_2,z_2,t_2) are world points and s is the distance between them.

For complete equivalence with the Euclidean distance, we need to have similar signs in the equation and get rid of the constant c. To that end,

$$\text{Let } \tau = ict\text{, such that } t = \frac{\tau}{ic} \qquad \text{Eqn (3)}$$

continued…

We obtain the following equation after making the substitution for t in Eqn (2):

$$s^2 = c^2\left(\frac{\tau_2}{ic} - \frac{\tau_1}{ic}\right)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$

$$\Rightarrow \quad s^2 = -(\tau_2 - \tau_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$

The equation for s now looks similar to that for L. For simplicity,

$Let\ y_2 = y_1\ and\ z_2 = z_1$. Then,

$$\Rightarrow \quad s^2 = -(\tau_2 - \tau_1)^2 - (x_2 - x_1)^2$$

Since s is invariant, we expect a transformation similar to that shown in Eqn (1) to switch between rotated space-time (τx) coordinate systems:

$$\begin{aligned} x &= x'\cos\theta - \tau'\sin\theta \\ \tau &= \tau'\cos\theta + x'\sin\theta \end{aligned} \qquad \text{Eqn (4)}$$

Intuition tells us that these transformations will preserve the invariance of s, just as the rotation transformations for Euclidean space, as given in Eqn (1), preserve distance.

Since the Lorentz transformations are valid in Minkowsky 4-dimensional space, we expect that the Lorentz transformations can be understood as rotations (such as those given in Eqn (4)) in such a space.

The Lorentz transformations are:

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad , \quad t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad , \quad y = y' \quad , \quad z = z'$$

With the substitution $t = \frac{\tau}{ic}$ (from Eqn (3)), the Lorentz transformations are:

$$x = \frac{x' + \frac{v}{ic}\tau'}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{x' - i\frac{v}{c}\tau'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \{note : \frac{1}{i} = -i\} \qquad \text{Eqn (5)}$$

and similarly,

$$\frac{\tau}{ic} = \frac{\frac{\tau'}{ic} + \frac{v}{c^2}x'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \Rightarrow \quad \tau = \frac{\tau' + i\frac{v}{c}x'}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{Eqn (6)}$$

continued…

Equations (5) and (6) can be rewritten as follows:

$$x \;=\; x'\left(\frac{1}{\sqrt{1-v^2/c^2}}\right) - \tau'\left(\frac{iv/c}{\sqrt{1-v^2/c^2}}\right) \qquad \text{Eqn (7)}$$

$$\text{and,}\;\; \tau \;=\; \tau'\left(\frac{1}{\sqrt{1-v^2/c^2}}\right) + x'\left(\frac{iv/c}{\sqrt{1-v^2/c^2}}\right) \qquad \text{Eqn (8)}$$

Note that: $(1)^2 + \left(i\frac{v}{c}\right)^2 = \left(\sqrt{1-\frac{v^2}{c^2}}\right)^2$, which can be interpreted as the Pythagorean relation for the following triangle:

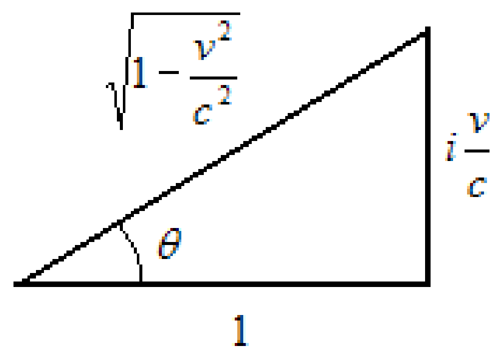

Figure 2.

From figure 2, $\tan\theta = i\frac{v}{c}$, $\quad \sin\theta = \dfrac{i\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$, $\quad$ *and* $\quad \cos\theta = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

Substituting these relations in Equations (7) and (8), we get:

$$x = x'\cos\theta - \tau'\sin\theta$$
$$\tau = \tau'\cos\theta + x'\sin\theta$$

These are exactly the rotation transformations that were anticipated.

To further simplify, note that $\tau = ict$ (from Eqn (3)) and $\theta = \tan^{-1}\left(i\frac{v}{c}\right)$. Making this substitution in the equations above:

$$x = x'\cos\theta - ict'\sin\theta = x'\cos\left(\tan^{-1}\left(i\frac{v}{c}\right)\right) - ict'\sin\left(\tan^{-1}\left(i\frac{v}{c}\right)\right)$$

continued…

and,

$$ict \;=\; ict'\cos\theta + x'\sin\theta \;=\; ict'\cos\left(\tan^{-1}\left(i\frac{v}{c}\right)\right) + x'\sin\left(\tan^{-1}\left(i\frac{v}{c}\right)\right)$$

$$\Rightarrow \quad t \;=\; t'\cos\left(\tan^{-1}\left(i\frac{v}{c}\right)\right) + \frac{x'}{ic}\sin\left(\tan^{-1}\left(i\frac{v}{c}\right)\right) \qquad \left\{\textit{multiplying both sides by } \frac{1}{ic}\right\}$$

Recall: $\sinh(x) = -i\sin(ix)$, $\cosh(x) = \cos(ix)$, $\tanh(x) - i\tan(ix)$. Using these, the above transformations can be reduced to:

$$x = x'\cosh\theta + t'\sinh\theta$$
$$t = t'\cosh\theta - x'\sinh\theta$$

Note: This result makes sense because the angle θ in figure 2 depends only on magnitude of v, which is the relative velocity of frame K' with respect to K. Angle θ does *not* depend on x or τ. Thus, the transformations preserve homogeneity of space and time, and isotropic property of space. Lorentz transformations are therefore equivalent to rotations in 4-D Minkowsky space - rotations with an angle that depends on the relative velocity v according to the relation: $\theta = \tan\left(i\frac{v}{c}\right)$.

3. The Lorentz transformations expressed as rotations preserve the interval $s^2 = c^2\Delta t^2 - \Delta x^2$. We know the following similar invariant relation between relativistic momentum and energy:

$$E^2 - c^2 p^2 = m_0^2 c^4$$

This is similar to the invariance of space-time interval *s*. Thus, Lorentz transformations for energy and momentum should exist (for switching between $E \leftrightarrow E', p'$, and $p \leftrightarrow E', p'$. Furthermore, these should be similar to Lorentz transformations for position and time.

We know that:

$$E = \frac{m_0 c^2}{\sqrt{1-\frac{u^2}{c^2}}}, \quad and \quad p = \frac{m_0 u}{\sqrt{1-\frac{u^2}{c^2}}} \text{ in frame K} \qquad \text{Eqn (1)}$$

$$E' = \frac{m_0 c^2}{\sqrt{1-\frac{u'^2}{c^2}}}, \quad and \quad p' = \frac{m_0 u'}{\sqrt{1-\frac{u'^2}{c^2}}} \text{ in frame K'}. \qquad \text{Eqn (2)}$$

Transformation for energy can be obtained as follows:

$$E = \frac{m_0 c^2}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{m_0 c^2}{\sqrt{1-\left(\frac{u'+v}{1+vu'/c^2}\right)/c^2}} \qquad \left\{substituting\ u = \frac{u'+v}{1+vu'/c^2}\right\}$$

$$= \frac{m_0 c^3\left(1+\frac{vu'}{c^2}\right)}{\sqrt{\left(1+\frac{vu'}{c^2}\right)c^2 - (u'+v)^2}} = \frac{m_0 c^3\left(1+\frac{vu'}{c^2}\right) \cdot \frac{1}{c}}{\sqrt{c^2 + \frac{v^2u'^2}{c^2} + 2u'v - u'^2 - v^2 - 2u'v} \cdot \frac{1}{c}}$$

$$= \frac{m_0 c^2\left(1+\frac{vu'}{c^2}\right)}{\sqrt{1+\frac{v^2u'^2}{c^4} - \frac{u'^2}{c^2} - \frac{v^2}{c^2}}} = \frac{m_0 c^2\left(1+\frac{vu'}{c^2}\right)}{\sqrt{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}}$$

$$= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{m_0 c^2 + \frac{vu' m_0 c^2}{c^2}}{\sqrt{1-\frac{u'^2}{c^2}}}\right) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{m_0 c^2}{\sqrt{1-\frac{u'^2}{c^2}}} + \frac{m_0 u'}{\sqrt{1-\frac{u'^2}{c^2}}} v\right)$$

$$\Rightarrow \quad E = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(E' + vp') \qquad \{substituting\ for\ E'\ and\ p'\ from\ Eqn\ (1)\ and\ (2)\}$$

continued…

The transformation for momentum is also similarly obtained:

$$p = \frac{m_0 u}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{m_0\left(\frac{u'+v}{1+vu'/c^2}\right)}{\sqrt{1-\left(\frac{u'+v}{1+vu'/c^2}\right)^2/c^2}} \qquad \left\{ substituting\ u = \frac{u'+v}{1+vu'/c^2} \right\}$$

$$= \frac{m_0 c\left(\frac{u'+v}{1+vu'/c^2}\right)\left(1+\frac{vu'}{c^2}\right) \ . \ \frac{1}{c}}{\sqrt{\left(1+\frac{vu'}{c^2}\right)^2 c^2-(u'+v)^2} \ . \ \frac{1}{c}} = \frac{m_0\left(\frac{u'+v}{1+vu'/c^2}\right)\left(1+\frac{vu'}{c^2}\right)}{\sqrt{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}}$$

$$= \frac{m_0 c^2\left(\frac{u'+v}{c^2+vu'}\right)\frac{\left(c^2+vu'\right)}{c^2}}{\sqrt{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}} = \frac{m_0(u'+v)}{\sqrt{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}}$$

$$= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{m_0 u'}{1-\frac{u'^2}{c^2}}+\frac{m_0 v}{1-\frac{u'^2}{c^2}}\right) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{m_0 u'}{1-\frac{u'^2}{c^2}}+v\left(\frac{m_0 c^2}{1-\frac{u'^2}{c^2}}\right)\frac{1}{c^2}\right)$$

$$\Rightarrow \ p = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(p'+\frac{vE'}{c^2}\right) \qquad \{ substituting\ for\ E'\ and\ p'\ from\ Eqn\ (1)\ and\ (2) \}$$

Thus, the Lorentz transformations for energy (E) and momentum (p) are:

$$E = \gamma\left(E'+vp'\right)$$
$$p = \gamma\left(p'+vE'/c^2\right)$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

References

The Classical Theory of Fields by L.D. Landau and E.M. Lifshitz.

Six Not so Easy Pieces by Richard Feynman.