PHYS 6107. Spring 2002.

HOMEWORK #5 $\sum = 100\ pts.$

<u>DUE:</u> Apr. 25 by 6 p.m. (in Dr. Marchenkov's mail box)

This homework is devoted to interacting systems, more specifically, to the Ising model of phase transitions in interacting systems.

Please, express the answers in terms of hyperbolic trigonometric functions (e.g., tanh(x), etc.)

Problem I.1 (25 pts)

A lattice of $N+1$ sites has spins $\sigma_i = \pm 1$ at each site, all of which are acted on by a magnetic field. There are interactions of equal strength between one of the spins, σ_0, and each of the others. Thus the Hamiltonian is

$$H = -h\sum_{i=0}^{N}\sigma_i - J\sum_{i=1}^{N}\sigma_i\sigma_0$$

1. Find the canonical partition function $Z(T,N)$, the average energy $\langle E\rangle$ and, for $i \neq 0$, the statistical averages $\langle\sigma_i\rangle$ and $\langle\sigma_0\sigma_i\rangle$. Compute the limits of these averages when $h \to 0$ with $J \neq 0$ and when $J \to 0$ with $h \neq 0$.
2. When $h = 0$, show, for $i, j \neq 0$ and $i \neq j$, that $\langle\sigma_i\sigma_j\rangle = \langle\sigma_0\sigma_i\rangle\langle\sigma_0\sigma_j\rangle$. Discuss the physical meaning of the last relation.

Problem I.2 (20 pts)

1. Spins on a one-dimensional lattice have a three-level Ising Hamiltonian, which in the absence of any external field is

$$H = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1}, \qquad \sigma_i = 1, 0, -1 \qquad J > 0$$

   Obtain the exact partition function in terms of temperature and the number of sites. Analyze the low-temperature limit of the internal energy for $N \gg 1$.
2. A "toy" model for mixtures of ^{3}He in ^{4}He is based on the following three-level spin-1 Hamiltonian for N spins on a lattice:

$$H = -J\sum_{i,j}^{nn}\sigma_i\sigma_j + K\sum_{i=1}^{N}\sigma_i^2 \qquad \sigma_i = 1, 0, -1$$

   Applying the mean-field approximation, derive the "self-consistency" equation for the mean spin $\bar{\sigma}$.

Problem I.3 (25 pts)

For a one-dimensional Ising model of N spins, with periodic boundary conditions, evaluate the correlation function for two nearest-neighbor spins:

$$G(j, j+1) = \langle \sigma_j \sigma_{j+1} \rangle - \langle \sigma_j \rangle \langle \sigma_{j+1} \rangle$$

in the limit $N \to \infty$. Investigate the special cases $J \to 0$ (non-interacting spins) and $B \to 0$ (no applied magnetic field). Discuss the physical meaning of these cases.

Problem I.4 (30 pts)

Consider a 2D Ising system (with magnetic field B and coupling $J > 0$) on a square lattice and investigate improvements to the mean-field approximation in the following way.

1. Find the effective field seen by a *pair* of nearest-neighbor spins, when the remaining spins are replaced by their mean values. Use this effective field to obtain a self-consistent equation for the mean value of each spin and calculate the quantity $\beta_c J$, where β_c is the inverted critical temperature. Compare this with the value obtained from the usual mean-field approximation.
2. Now find the effective field seen by the spins in an entire row of lattice sites. For a 1D Ising model, the exact mean value of each spin is $\bar{\sigma} = \sinh(\beta\mu B)\left[\sinh^2(\beta\mu B) + e^{-4\beta J}\right]^{-1/2}$. Use this fact to obtain an improved estimate of $\beta_c J$.
3. Compare the approximations of $\beta_c J$ found in (1) and (2) with the exact value obtained by Onsager. Discuss the physical meaning.