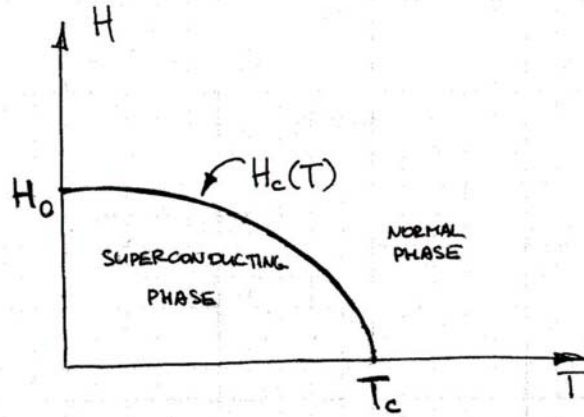


MIDTERM. PHYS 6107. Spring 2002.

Problem I (25 pts)



Consider a type I superconducting material with a parabolic coexistence curve separating the uniform Superconducting and normal phases (see figure). H is the external magnetic field, and T is the temperature. Ignore the tiny magnetization of the normal phase. The critical field H_c as a function of temperature is given by

$$H_c(T) = H_0 + aT + bT^2$$

1. (5 pts) Why must the coefficient a be zero?

The first law of thermodynamics for a material in a magnetic field can be written as

$$d\varepsilon = Tds + Hdm,$$

where ε, s, m are the energy, entropy, and the magnetic moment per mole, respectively. In the equation above, the member $+Hdm$ substitutes the member $-Pdv$ in the conventional expression for the first law of thermodynamics. The analogous Clayperon equation is thus

$$\frac{dH_c}{dT} = \frac{s_n - s_s}{m_s - m_n} = -\frac{\Delta s}{\Delta m}.$$

From the 3rd law of thermodynamics we must have $s \rightarrow 0$ in both phases as $T \rightarrow 0$, which means that $\Delta s \rightarrow 0$.

$$\boxed{\therefore \frac{dH_c}{dT} \rightarrow 0 \quad \text{as} \quad T \rightarrow 0 \Rightarrow a = 0}$$

2. (10 pts) Calculate the latent heat per unit volume as a function of T along the coexistence curve in terms of H_0 and T_c , as shown in the figure.

The molar latent heat can be expressed as $l \equiv T\Delta s$. Therefore, the Clayperon equation can be written as

$$\frac{dH_c}{dT} = -\frac{l}{T\Delta m}.$$

This equation will be the same if the extensive quantities are per unit volume rather than per mole. Taking the derivative and remembering that $a = 0$, we have

$$l = -2bT^2\Delta m = -2bT^2(m_s - m_n).$$

In order to find b , consider H_c when $T = T_c$:

$$H_c = 0 = H_0 + bT_c^2$$

$$\therefore \boxed{b = -\frac{H_0}{T_c^2}}$$

Thus

$$l = 2H_0\left(\frac{T}{T_c}\right)^2 \Delta m = 2H_0\left(\frac{T}{T_c}\right)^2 m_s.$$

Inside a type I superconductor the magnetic field $\vec{B} = 0$ (Meissner effect). Thus $\vec{B} = \vec{H} + 4\pi\vec{m} = 0$, so that

$$m_s = -\frac{H_c}{4\pi}$$

Finally, we can obtain the latent heat going from the normal to the superconducting state:

$$\boxed{l = -\frac{H_0^2}{2\pi}\left(\frac{T}{T_c}\right)^2\left[1 - \left(\frac{T}{T_c}\right)^2\right]}$$

3. (10 pts) Calculate the discontinuity in the specific heat per unit volume at constant H along the coexistence curve.

[Hints: Inside the superconducting phase, the magnetic field $B = 0$. The Gibbs free energy in a magnetic field H is given by $G = E - TS - HM$, where M is the magnetization.]

The discontinuity in the specific heat per unit volume can be expressed as

$$\Delta c = T\left[\left(\frac{\partial s_s}{\partial T}\right)_H - \left(\frac{\partial s_n}{\partial T}\right)_H\right] = T\left(\frac{\partial \Delta s}{\partial T}\right)_H = T\left(\frac{\partial(l/T)}{\partial T}\right)_H$$

Therefore

$$\boxed{\Delta c = -\frac{H_0^2}{2\pi}\frac{T}{T_c^2}\left[1 - 3\left(\frac{T}{T_c}\right)^2\right]}$$

Problem 2 (25 pts)

Consider a vapor (dilute gas of atoms of mass m) in equilibrium with a submonolayer (i.e. less than one atomic layer) adsorbed film on a plane surface at temperature T . When atoms are adsorbed, their energy is $-\varepsilon$ per atom. Assume there is a large number N of possible sites for adsorption. Find the vapor pressure as a function of surface concentration, n/N , where n is the number of adsorbed atoms.

The equilibrium condition implies that the chemical potentials of both phases are equal. We need to find both of them and obtain the vapor pressure. In general, the chemical potential is defined as

$$\mu = \frac{\partial F(N, T)}{\partial N} = -\frac{\partial (T \ln Z)}{\partial N}$$

The result for the partition function for one particle in the vapor is (ideal gas):

$$Z_1 = \frac{V}{(2\pi\hbar)^3} \left[\int e^{-\frac{p_x^2}{2mT}} dp_x \right]^3 = \frac{V}{(2\pi\hbar)^3} (2\pi mT)^{\frac{3}{2}}$$

Thus the total partition function for the gas is

$$Z_g = \frac{1}{N_g!} Z_1^{N_g},$$

where N_g is the number of particles in the vapor. Solving for the chemical potential (using Stirling's approximation), we find

$$\mu_g \approx -T \ln \left[\frac{V_g}{N_g} \left(\frac{mT}{2\pi\hbar^2} \right)^{\frac{3}{2}} \right] = \underbrace{-T \ln \left[\frac{T}{p_g} \left(\frac{mT}{2\pi\hbar^2} \right)^{\frac{3}{2}} \right]}_{PV=NT}$$

Consider the adsorbed phase. The number of possible states is

$$g = \frac{N!}{(N-n)!n!}$$

and the partition function is

$$Z = \frac{N!}{(N-n)!n!} e^{\frac{n\varepsilon_0}{T}}.$$

As usual, the chemical potential is obtained from the calculation of the mean number of adsorbed particles:

$$\langle n \rangle = T \frac{\partial}{\partial \mu_s} (\ln \Xi) = \frac{T}{\Xi} \frac{\partial \Xi}{\partial \mu_s},$$

$$\Xi = \sum_{n=0}^N z^n Z = \sum_{n=0}^N e^{\frac{n\mu_s}{T}} Z = \left(1 + e^{\frac{\mu_s + \varepsilon_0}{T}} \right)^N$$

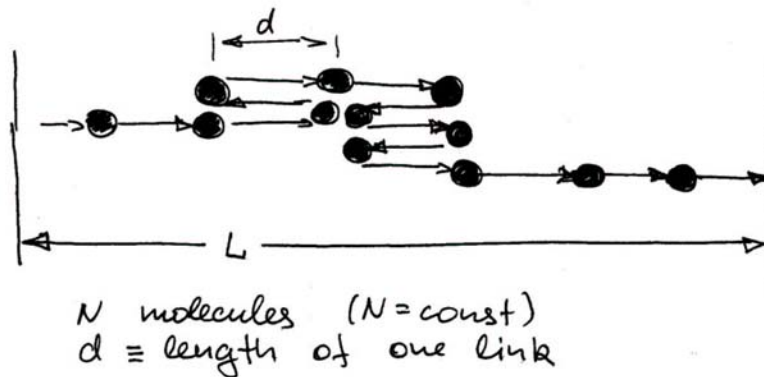
We can then obtain

$$\mu_s = -T \ln \left[\left(\frac{N}{\langle n \rangle} - 1 \right) e^{\frac{\epsilon_0}{T}} \right]$$

Upon equating the chemical potentials and taking $n = \langle n \rangle$, we find

$$p_g = \left(\frac{mT}{2\pi\hbar^2} \right)^{\frac{3}{2}} \frac{T}{N/n - 1} e^{-\frac{\epsilon_0}{T}}$$

Problem 3 (25 pts)



The elasticity of a rubber band can be described in terms of a one-dimensional model of polymer involving N molecules linked together end-to-end (figure). The angle between links is equally likely to be 0° or 180° .

- (8 pts) Find the number of arrangements that give an overall length of $L = 2md$, where m is positive. Indicate clearly the reasoning you used to get this result.

Assume that the polymer chain contains N_+ links of 0° angle and N_- links of 180° angle. Then:

$$N_+ - N_- = 2m \quad \text{and} \quad N_+ + N_- = N.$$

$$\therefore \begin{cases} N_+ = \frac{N}{2} + m \\ N_- = \frac{N}{2} - m \end{cases}$$

This gives us $\frac{N!}{N_+!N_-!}$ arrangements. Note the asymmetry in this system, namely $N_+ > N_-$. But the same length can be obtained if $N_- > N_+$; that is the angles are reversed for every arrangement (we are having then $N_- - N_+ = 2m$), the overall length is still $2md$. Thus

$$g(N, m) = 2 \frac{N!}{\left(\frac{N}{2} + m\right)! \left(\frac{N}{2} - m\right)!}$$

2. (4 pts) Find the entropy of the system as a function of L for $N \gg 1, L \ll Nd$.

The condition $L \ll Nd$ means $2md \ll Nd \Rightarrow m \ll N$. In this limit we can obtain

$$g(N, m) \approx g(N, 0) e^{-\frac{2m^2}{N}}$$

The entropy of the system becomes

$$\begin{aligned} S &= \ln g(N, m) = \ln g(N, 0) - \frac{L^2}{2Nd^2} = \\ &= \ln 2 + \ln N! - 2 \ln \left(\frac{N}{2}\right)! - \frac{L^2}{2Nd^2} \approx \\ &\approx (N + 1) \ln 2 - \frac{L^2}{2Nd^2} \end{aligned}$$

3. (6 pts) Find the force required to maintain the length L for $L \ll Nd$.

From the thermodynamic relations $dE = TdS + fdL$ and $F = E - TS$ we obtain $dF = -SdT + fdL$. Therefore

$$\begin{aligned} \left(\frac{\partial f}{\partial T}\right)_L &= -\left(\frac{\partial S}{\partial L}\right)_T = \frac{L}{Nd^2} \\ \therefore f &= \frac{TL}{Nd^2} + \text{const} \end{aligned}$$

Since $f = 0$ when $L = 0$

$$f = \frac{TL}{Nd^2}$$

4. (7 pts) Find the relation between the force and the length without restriction in (3), i.e. for all possible values of L ; maintain, however, $N \gg 1$.

Consider a single link. When the external force is exerted, the probability for the link to have a certain angle is:

$$\begin{cases} 0^\circ & \Leftrightarrow e^{\frac{fd}{T}} \\ 180^\circ & \Leftrightarrow e^{-\frac{fd}{T}} \end{cases}$$

The average length of a link is

$$\langle l \rangle = \frac{e^{\frac{fd}{T}} - e^{-\frac{fd}{T}}}{e^{\frac{fd}{T}} + e^{-\frac{fd}{T}}} d = d \tanh\left(\frac{fd}{T}\right)$$

The overall length is then

$$L = N \langle l \rangle = Nd \tanh\left(\frac{fd}{T}\right)$$

Problem 4 (25 pts)

A system of two energy levels $E_1 > E_0 \neq 0$ is populated by N particles at temperature T . The particles populate the energy levels according to the classical (Boltzmann) distribution law.

1. (12 pts) Derive an expression for the average energy per particle and evaluate it at $T \rightarrow 0$ and $T \rightarrow \infty$.

The average energy of a particle is:

$$\frac{\langle E \rangle}{N} = \frac{E_0 e^{-\beta E_0} + E_1 e^{-\beta E_1}}{e^{-\beta E_0} + e^{-\beta E_1}}$$

Letting $\Delta E = E_1 - E_0$ we get

$$\frac{\langle E \rangle}{N} = \frac{E_0 + E_1 e^{-\beta \Delta E}}{1 + e^{-\beta \Delta E}}$$

When

$$T \rightarrow 0 \Leftrightarrow \beta \rightarrow \infty$$

$$\therefore \frac{\langle E \rangle}{N} \approx (E_0 + E_1 e^{-\beta \Delta E})(1 - e^{-\beta \Delta E}) = \boxed{E_0 + \Delta E e^{-\beta \Delta E}}$$

When

$$T \rightarrow \infty \Leftrightarrow \beta \rightarrow 0$$

$$\therefore \frac{\langle E \rangle}{N} \approx \frac{1}{2}(E_0 + E_1 - \beta E_1 \Delta E) \left(1 + \frac{1}{2} \beta \Delta E\right) \approx \boxed{\frac{1}{2}(E_0 + E_1) - \frac{\beta}{4}(\Delta E)^2}$$

2. (13 pts) Derive an expression for the specific heat of the system of N particles and evaluate it at $T \rightarrow 0$ and $T \rightarrow \infty$.

The specific heat for N particles is

$$C = \frac{\partial \langle E \rangle}{\partial T} = \frac{\partial \langle E \rangle}{\partial \beta} \frac{\partial \beta}{\partial T} = -\beta^2 \frac{\partial \langle E \rangle}{\partial \beta} =$$

$$= \boxed{\beta^2 N (\Delta E)^2 \frac{e^{-\beta \Delta E}}{(1 + e^{-\beta \Delta E})^2}}$$

When

$$T \rightarrow 0 \Leftrightarrow \beta \rightarrow \infty$$

$$\boxed{C \approx \beta^2 N (\Delta E)^2 e^{-\beta \Delta E}}$$

When

$$T \rightarrow \infty \Leftrightarrow \beta \rightarrow 0$$

$$\boxed{C \approx \frac{\beta^2 N (\Delta E)^2}{4}}$$