(a) As usual, we quantize the system in a 3D box of volume V. Each single particle state is characterized by its momentum $\bar{p}$ and the integer n. The grand canonical partition function is therefore a product of partition sums for single-particle states labelled by both $\bar{p}$ and n so that the pressure and number density are:

$$P = \frac{T}{V} \ln \Xi = -\frac{T}{V} \sum_{\bar{p}} \sum_{n=-j}^{j} \ln\left(1 - z e^{-\beta \alpha n} e^{-\beta \varepsilon(\bar{p})}\right)$$

$$\frac{N}{V} = \frac{z}{V} \frac{\partial \ln \Xi}{\partial z} = \frac{1}{V} \sum_{\bar{p}} \sum_{n=-j}^{j} \left(\frac{1}{z} e^{\beta \alpha n} e^{\beta \varepsilon(\bar{p})} - 1\right)^{-1}$$

where $\varepsilon(p) = \frac{|\bar{p}|^2}{2m}$, z is the fugacity.

In the T-limit, the sum over momenta can be replaced by an integral. However (similar to the discussion in class) to allow for a possible macroscopic occupation of the ground state, the corresponding term in the sum for $\frac{N}{V}$ (but not in the sum for P!) must be explicitly retained. The lowest energy state is that with $|\bar{p}|=0$ and $n=-j$

$$\therefore \quad P = T\left(\frac{2\pi m T}{(2\pi\hbar)^2}\right)^{3/2} \sum_{n=-j}^{j} g_{5/2}\left(z e^{-\beta \alpha n}\right)$$

$$\frac{N}{V} = \left(\frac{2\pi m T}{(2\pi\hbar)^2}\right)^{3/2} \sum_{n=-j}^{j} g_{3/2}\left(z e^{-\beta \alpha n}\right) + \frac{1}{V} \frac{z e^{\beta \alpha j}}{1 - z e^{\beta \alpha j}}$$

(b) The ground state becomes macroscopically occupied when z approaches its maximum value $z_{max} = e^{-\beta\alpha j}$.

More specifically, in the T-limit we set

$$z_{\cancel{max}} \approx z_{max} - \frac{1}{n_0 V}$$

and take the limit $V \to \infty$ with $\frac{N}{V}$ fixed. The critical temperature T_c can then be identified by setting $n_0 = 0$ in the remaining equation for $\frac{N}{V}$. With the change of the summation variable $\sigma = n + j$:

$$\frac{N}{V} = \left(\frac{2\pi m T_c}{(2\pi\hbar)^2}\right)^{3/2} \sum_{\sigma=0}^{2j} g_{3/2}\left(e^{-\frac{\alpha\sigma}{T_c}}\right)$$

- This is an implicit relation for T_c.

if $\alpha = 0$, we can solve this equation:

$$T_c = (2j+1)^{-\frac{2}{3}} T_c^0$$

This is the critical temperature for spin-j bosons expressed through the critical temperature of spin-0 bosons, T_c^0.