ⓐ The energy of black body radiation is

$$E = 2 \iint \frac{d^n p\, d^n q}{(2\pi\hbar)^n} \frac{\hbar\omega}{e^{\frac{\hbar\omega}{T}} - 1} =$$

$$= \frac{2V}{(2\pi\hbar)^n} \int d^n p \frac{\hbar\omega}{e^{\frac{\hbar\omega}{T}} - 1}$$

For photons we have $p = \frac{\hbar\omega}{c}$ ← speed of light

$$\therefore \frac{E}{V} = 2\left(\frac{1}{2\pi\hbar c}\right)^n \int d^n x \frac{x}{e^x - 1} T^{n+1}$$

$$\therefore \underline{\underline{\alpha = n+1}}$$

ⓑ The Debye model regards solids as an isotropic continuous medium with the partition sum:

$$Z(T,V) = \exp\left[-\hbar \sum_{i=1}^{nN} \frac{\omega_i}{2T}\right] \prod_{j=1}^{nN} \left[1 - \exp\left(\frac{\hbar\omega_j}{T}\right)\right]^{-1}$$

The Helmholtz free energy

$$F = -T \ln Z = \frac{\hbar}{2} \sum_{i=1}^{nN} \omega_i + T \sum_{j=1}^{nN} \ln\left[1 - \exp\left(\frac{\hbar\omega_j}{T}\right)\right]$$

When N is large,

$$\sum_{i=1}^{nN} \rightarrow \frac{n^2 N}{\omega_D^n} \int_0^{\omega_D} \omega^{n-1}\, d\omega$$

$$\therefore F = \frac{n^2 N}{2(n+1)} \hbar\omega_D + T^{n+1} \frac{n^2 N}{(\hbar\omega_D)^n} \int_0^{x_D} x^{n-1} \ln[1 - \exp(-x)]\, dx$$

where $x_D = \frac{\hbar\omega_D}{T}$

$$\therefore c_V = -\frac{1}{\beta}\left(\frac{\partial^2 F}{\partial T^2}\right) \propto T^n, \text{ i.e. } \underline{\underline{\beta = n}}$$

(c) The equipartition theorem gives the constant volume heat capacity of a molecule as $c_v = \frac{l}{2}$, where l is the number of the degrees of freedom. For a monatomic molecule in an n-dimensional space $l = n$. $\Rightarrow$ $c_p = c_v + 1$ $\Rightarrow$

$$\gamma = \frac{c_p}{c_v} = \frac{n+2}{n}$$