We begin by writing the grand canonical partition function as a sum over multi-particle states specified by occupation numbers of single-particle states. For bosons and fermions:

$$\cancel{Z}\,\Xi = \sum_{\{n_i\}} \exp\left(-\beta \sum_i n_i(\varepsilon_i - \mu)\right) = \prod_i \sum_{n_i} e^{-\beta(\varepsilon_i-\mu)n_i}$$

while for Maxwell-Boltzmann particles:

$$\Xi = \prod_i \sum_{n_i} \frac{1}{n_i!} e^{-\beta(\varepsilon_i-\mu)n_i}$$

indistinguishable particles

In each case, we see that Ξ has the form $\prod_i L_i$, where L_i can be regarded as the partition function for the occupancy of the ith single-particle state. Consequently, we can pick out the probabilities

$$w_i(n_i, T, \mu) = \begin{cases} \dfrac{1}{L_i} e^{-\beta(\varepsilon_i-\mu)n_i} & \text{- BE \& FD particles} \\[2ex] \dfrac{1}{L_i} \dfrac{e^{-\beta(\varepsilon_i-\mu)n_i}}{n_i!} & \text{- MB particles} \end{cases}$$

with

$$L_i = \sum_{n_i} e^{-\beta(\varepsilon_i-\mu)n_i} = \left(1 \pm e^{-\beta(\varepsilon_i-\mu)}\right)^{\pm 1}$$

$+$ = Fermions
$-$ = Bosons

and

$$L_i = \sum_{n_i} \frac{1}{n_i!} e^{-\beta(\varepsilon_i-\mu)n_i} = \exp\left(e^{-\beta(\varepsilon_i-\mu)}\right)$$

for MB particles.

The average occupation numbers are

$$\langle n_i \rangle = \sum_{n_i} n_i w_i = \frac{\partial L_i}{\partial(\beta\mu)} = \begin{cases} \dfrac{1}{e^{\beta(\varepsilon_i - \mu)} \pm 1} & \text{FD \& BE} \\ e^{-\beta(\varepsilon_i - \mu)} & \text{MB} \end{cases}$$

From this, we get

$$w_i^{BE} = \frac{1}{1+\langle n_i \rangle}\left(\frac{\langle n_i \rangle}{1+\langle n_i \rangle}\right)^{n_i}$$

$$w_i^{FD} = (1-\langle n_i \rangle)\left(\frac{\langle n_i \rangle}{1-\langle n_i \rangle}\right)^{n_i}$$

$$w_i^{MB} = \frac{e^{-\langle n_i \rangle}\langle n_i \rangle^{n_i}}{n_i!}$$

The variance of the distribution is

$$(\Delta n_i)^2 = \sum_{n_i} n_i^2 w_i - \left(\sum_{n_i} n_i w_i\right)^2 = \frac{\partial^2 L_i}{\partial(\beta\mu)^2}$$

And we easily find:

$$\left(\frac{\Delta n_i}{\langle n_i \rangle}\right)_{BE} = \sqrt{\frac{1}{\langle n_i \rangle} + 1}$$

$$\left(\frac{\Delta n_i}{\langle n_i \rangle}\right)_{FD} = \sqrt{\frac{1}{\langle n_i \rangle} - 1}$$

$$\left(\frac{\Delta n_i}{\langle n_i \rangle}\right)_{MB} = \sqrt{\frac{1}{\langle n_i \rangle}}$$.