The Hamiltonian for a 3D oscillator is simply the sum of Hamiltonians for 3 1D; so we are free to regard the system as a collection either of 3N 1D oscillators or N 3D oscillators. In either case, since the oscillators do not interact, the partition function Z is a product of the partition functions of the individual oscillators:

$$Z(T,N) = \prod_{i=1}^{M} Z_i(T) = \prod_{i=1}^{M} \left(\sum_{n_i=0}^{\infty} g(n_i) e^{-\beta E_i(n_i)} \right)$$

where M is either 3N or N and, for the ith oscillator, $g(n_i)$ is the degeneracy of the energy level labelled by n_i.

(a) For a 1D oscillator of angular frequency ω, the energy levels are $E_i(n_i) = (n_i + \frac{1}{2})\hbar\omega$ and are non-degenerate, i.e. $g(n_i) = 1$.

$$\therefore \; Z(T,N) = \prod_{i=1}^{3N} \left(\sum_{n_i=0}^{\infty} e^{-\beta\hbar\omega(n_i+\frac{1}{2})} \right) = \left(e^{-\frac{\beta\hbar\omega}{2}} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} \right)^{3N}$$

Since $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ if $q = e^{-\beta\hbar\omega} < 1$

$$\Downarrow$$

$$Z(T,N) = \left(\frac{e^{-\frac{\beta\hbar\omega}{2}}}{1-e^{-\beta\hbar\omega}} \right)^{3N}$$

(b) The energy levels of an isotropic 3D oscillator are $E_i(n_i) = (n_i + \frac{3}{2})\hbar\omega$. The degeneracy of one of these levels is the number of sets of three non-negative integers that add to n_i. This can be found, for example, by the same method as distribution of balls in boxes.

$$g(n_i) = \frac{(n_i+2)!}{2!\,n_i!} = \frac{1}{2}n_i^2 + \frac{3}{2}n_i + 1$$

The partition function for one oscillator is

$$Z_i = e^{-3\beta\hbar\omega/2} \sum_{n_i=0}^{\infty} \left(\frac{1}{2}n_i^2 + \frac{3}{2}n_i + 1\right) e^{-\beta\hbar\omega n_i}.$$

Two additional relation required to evaluate this sum are:

$$\sum_{n=0}^{\infty} n q^n = q\frac{d}{dq}[L(q)] = \frac{q}{(1-q)^2}$$

$$\sum_{n=0}^{\infty} n^2 q^n = \left(q\frac{d}{dq}\right)^2 [L(q)] = \frac{q(1+q)}{(1-q)^3}$$

$$\therefore\ Z = \left[e^{-3\beta\hbar\omega/2}\left(\frac{1}{2}\frac{q(1+q)}{(1-q)^3} + \frac{3}{2}\frac{q}{(1-q)^2} + \frac{1}{1-q}\right)\right]^N$$

which, with $q = e^{-\beta\hbar\omega}$, can be reduced to the previous result.