

(a) First of all, obtain the "Maxwell relation" for the band (compare second derivatives):

$$\left[ \frac{\partial}{\partial T} \left( \frac{\partial S}{\partial L} \right)_T \right]_L = \left( \frac{\partial f}{\partial T} \right)_L = \left( \frac{\partial g}{\partial L} \right)_T = \left[ \frac{\partial}{\partial L} \left( \frac{\partial S}{\partial T} \right)_L \right]_T$$

! :  $g$  is independent of temperature :  $g = g(L)$

$$\int g \Rightarrow \mathcal{J}(T, L) = Tg(L) + h_1(L) \quad \leftarrow \text{unknown function}$$

$$\int f \Rightarrow \mathcal{J}(T, L) = Tg(L) + h_2(L), \text{ since}$$

$$g'(L) = \frac{f(T, L)}{T}$$

We argue then that  $h_1 = h_2$  and both are equal to zero, since the tension  $J$  must vanish when  $L = L_0$

$$\therefore \underline{\underline{\mathcal{J}(T, L) = T \frac{aL}{L_0} \left[ 1 - \left( \frac{L_0}{L} \right)^3 \right]}}$$

(b) The work is done on the band when its length increases:

$$\therefore dE = TdS + \mathcal{J}dL \quad (*)$$

The appropriate thermodynamic potential (analogous to the Helmholtz free energy)

$$F(T, L) = E - TS \text{ for which}$$

$$dF = -SdT + \mathcal{J}dL$$

The "Maxwell relation" here is

$$\left( \frac{\partial S}{\partial L} \right)_T = - \left( \frac{\partial \mathcal{J}}{\partial T} \right)_L$$

Calculate:

$$\left(\frac{\partial E}{\partial L}\right)_T = T \left(\frac{\partial S}{\partial L}\right)_T + J = -T \left(\frac{\partial J}{\partial T}\right)_L + J = 0,$$

$$\text{since } \left(\frac{\partial J}{\partial T}\right)_L = \left(\frac{\partial Tg(L)}{\partial T}\right)_L = \frac{J}{T}$$

Then,  $E$  depends on  $T$  only and

$$dE = C_L dT$$

$\therefore$  The first law, ( $x$ ), is:

$$dS = C_L \frac{dT}{T} - \frac{J}{T} dL = C_L \frac{dT}{T} - g(L) dL$$

$\Downarrow$

variables are separated and we can integrate between two states:

$$\Delta S = C_L \int_{T_1}^{T_2} \frac{dT}{T} - \int_{L_1}^{L_2} g(L) dL = C_L \ln\left(\frac{T_2}{T_1}\right) - a L_0 \left[ u\left(\frac{L_2}{L_0}\right) - u\left(\frac{L_1}{L_0}\right) \right],$$

$$\text{where } u(x) = \frac{x^2}{2} + \frac{1}{x}$$

This expression is independent on the route taken between the two states.

When the route is reversible and adiabatic from  $L_0 \rightarrow L_f$ :  $\Delta S = 0$

$$T_f = T_i \left[ \frac{1}{e_L} a L_0 \left[ u\left(\frac{L_f}{L_0}\right) - u\left(\frac{L_i}{L_0}\right) \right] \right]$$

since for  $x > 1$  ( $L_f > L_0$ )  $u(x)$  is an increasing function, the temperature of the band has increased!

© During the free contraction, no work is done and no heat is exchanged. Therefore,  $E$  and hence  $T$  are unchanged.

$$\therefore \Delta S = -aL_0 \left[ u(1) - u\left(\frac{L_f}{L_0}\right) \right] > 0$$

This is an analogy to the free expansion of an ideal gas (a.k.a. Joule effect).

$\Delta S > 0$  since the free contraction is an irreversible process.