Problem III.8 | Sheet 1

① The number of states with total energy $E = n_+ \varepsilon$ is

$$\Omega(E,N) = \binom{N}{n_+} g^{n_+} = \frac{N!\, g^{n_+}}{n_+!\,(N-n_+)!}$$

where the binomial coefficent counts the number of ways of choosing n_+ distinguishable particles to occupy the upper level, and g^{n_+} is the number of ways of assigning these particles tho the g degenerate states. For large N, we use Stirling's approximation to compute the entropy as

$$S(E,N) = \ln \Omega(E,N) \approx N[x\ln x + (1-x)\ln(1-x) - \ln g],$$

where $x = \frac{n_+}{N} = \frac{E}{N\varepsilon}$

The temperature is given:

$$\beta = \frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N = \frac{1}{N\varepsilon}\left(\frac{\partial S}{\partial x}\right)_N = \frac{1}{\varepsilon}\ln\left(\frac{g(1-x)}{x}\right)$$

Solving the relation for x, we find that:

$$\underline{\underline{n_+ = Nx = N\frac{g e^{-\beta\varepsilon}}{1+g e^{-\beta\varepsilon}}}} \; ; \; \underline{\underline{n_0 = N - n_+ = N\frac{1}{1+g e^{-\beta\varepsilon}}}}$$

② If $E = 0.75 N\varepsilon \Rightarrow x = 0.75$, and with $g=2$, we get $\beta\varepsilon = \ln\frac{2}{3} \approx -0.4$

Let us assume that a system has bounded energy, i.e. there is a value E_{max}, which is the greatest possible value.

It is very likely that for such a system, very few microstates correspond to both $E=0$ and $E = E_{max}$. $\Rightarrow$ S should be small in both cases.

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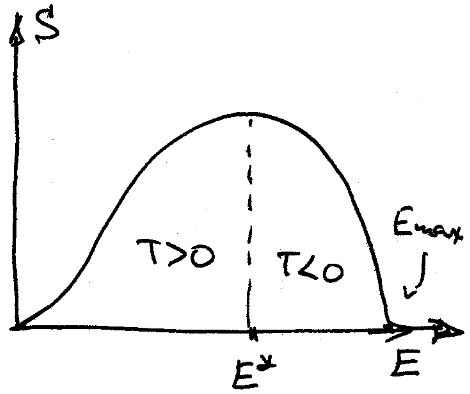
Thus, the entropy should have maximum at some intermediate value of energy, E^*. $\beta = \frac{1}{T} \equiv \frac{\partial S}{\partial E} > 0$ if $E < E^*$

$$\beta = \frac{1}{T} \equiv \frac{\partial S}{\partial E} < 0, \text{ if } E > E^*$$

Note: $E^* \neq \langle E \rangle$

Thus: a negative temperature state has greater energy than the positive temperature state, and is, thus, hotter.

Heat does not flow spontaneously from a cooler body to a hotter body => system cannot be "heated" to negative temperature by an inflow of heat, so long as its surroundings are at finite temperature. Heat will always flow from a negative-T system in contact with positive-T surroundings.



Let us check this conclusion from the point of view of entropy changes. If an amount ΔE of energy flows from the system into the heat bath, then the total entropy change is:

$$\Delta S = \Delta S|_{bath} + \Delta S|_{SYSTEM} = \left(\frac{\partial S}{\partial E}\Big|_{bath} - \frac{\partial S}{\partial E}\Big|_{SYSTEM} \right) \Delta E = \left(\frac{1}{T_{bath}} - \frac{1}{T_{SYSTEM}} \right) \Delta E.$$

For $\Delta S > 0$, ΔE must be > 0 if T^{-1}_{SYSTEM} is smaller than T^{-1}_{bath} and vice-versa. Clearly, for the purpose of calculating this sign, a negative value of T^{-1} counts as smaller than a positive value. Thus, from this point of view, we see that a negative T is higher than a positive T, and heat flows from T<0 system to the bath.