Prolem III.7 | Binary chain.

(a) Why not :-)

(b) Let's call w_+ and w_- the probabilities of given sites having energies $+\varepsilon$ an $-\varepsilon$ respectively.

$$w_+ = \frac{e^{-\beta\varepsilon}}{Z_1} \qquad w_- = \frac{e^{\beta\varepsilon}}{Z_1}$$

A given site belongs to a cluster of length L if:

1. it has energy $+\varepsilon$
2. for some n, $0 \le n \le L-1$, there are n sites to its left, and $(L-n-1)$ sites to its right which also have energy $+\varepsilon$
3. the sites $n+1$ places to its left and $(L-n)$ places to its right have energy $-\varepsilon$

The probability for this situation is

$$w_+^L w_-^2.$$

Thus the TOTAL probability that a site belongs to a cluster of length L is

$$w_L = \begin{cases} L w_+^L w_-^2 & , L \ge 1 \\ w_- & , L = 0 \end{cases}$$

For $L \ge 1$, the factor L accounts for the L possible values of n, while a cluster with $L=0$ is a site having energy $-\varepsilon$.
In the above, I ignored the possibility that the given site is within a distance L of the ends of the whole lattice, which would change probabilities. For a finite L, this is legitimate in the limit $N \to \infty$.

Note that:

$$\sum_{L=1}^{\infty} L w_+^L = w_+ \frac{\partial}{\partial w_+}\left(\sum_{L=0}^{\infty} w_+^L\right) = w_+ \frac{\partial}{\partial w_+}\left(\frac{1}{1-w_+}\right) =$$

$$= \frac{w_+}{(1-w_+)^2} = \frac{w_+}{w_-^2}$$

$$\therefore \quad \sum_{L=0}^{\infty} w_L = w_- + w_-^2 \sum_{L=1}^{\infty} L w_+^L = w_- + w_+ = 1.$$

With this interpretation of a cluster of zero length, any chosen site belongs to a cluster of some length.

ⓒ Let us first define what is the average length of a cluster (there are several ways, which are not necessarily equivalent).

<u>My definition</u>: choose a particular site on the lattice. For each state of the system, it belongs, as discussed above, to a cluster of <u>some</u> length L, and we average this length over all possible states of the system. In the limit $N \to \infty$, this $\langle L \rangle$ is independent on which site we choose.

$$\therefore \quad \langle L \rangle = \sum_{L=0}^{\infty} L \, \cancel{P} w_L = w_-^2 \sum_{L=1}^{\infty} L^2 w_+^L =$$

$$= w_-^2 w_+ \frac{\partial}{\partial w_+} \left(\sum_{L=1}^{\infty} L \, w_+^L \right) =$$

$$= w_-^2 w_+ \frac{\partial}{\partial w_+} \left(\frac{w_+}{(1-w_+)^2} \right) = \frac{w_+ (1+w_+)}{w_-}$$

In terms of β:

$$\langle L \rangle = e^{-2\beta\varepsilon} \, \frac{e^{\beta\varepsilon} + 2e^{-\beta\varepsilon}}{e^{\beta\varepsilon} + e^{-\beta\varepsilon}}$$

When $T=0$ we have $\langle L \rangle_{T=0} = 0$, i.e. all sites on the lattice are in the lower energy level $\Rightarrow$ every cluster has zero length.

When $T \to \infty$ $\quad \begin{matrix} w_+ \to \frac{1}{2} \\ w_- \to \frac{1}{2} \end{matrix} \Rightarrow \langle L \rangle_\infty = \frac{3}{2}$