PROBLEM III.4.

The position of a point particle on a sphere of radius r can be specified in spherical polar coordinates as

$$r = r\sin\theta\cos\varphi\,\hat{x} + r\sin\theta\sin\varphi\,\hat{y} + r\cos\theta\,\hat{z}$$

with $0 \leq \theta \leq \pi$, and $0 \leq \varphi \leq 2\pi$

We need to find the correct phase space variables. To do it, first consider the Lagrangian for a single particle:

$$L = \frac{1}{2}m|\dot{r}|^2 = \frac{1}{2}mr^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2)$$

The momenta conjugate to θ and φ are:

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mr^2\dot{\theta} \quad \text{and} \quad p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = mr^2\sin^2\theta\,\dot{\varphi}$$

The Hamiltonian then reads:

$$H(\theta, \varphi, p_\theta, p_\varphi) = \frac{1}{2mr^2}p_\theta^2 + \frac{1}{2mr^2}\frac{1}{\sin^2\theta}p_\varphi^2$$

Note: r is a geometrical constant, not a variable.

$$Z_1 = \int e^{-\beta H}\,d\theta\,d\varphi\,dp_\theta\,dp_\varphi =$$

one particle

$$= \int\sqrt{2\pi mr^2T}\sqrt{2\pi mr^2T\sin^2\theta}\,d\theta\,d\varphi =$$

$$= (2\pi mr^2T)2\pi\int_0^\pi|\sin\theta|\,d\theta = \underline{\underline{2\pi mAT}}$$

where $A = 4\pi r^2$ – the area of the sphere.

$$Z = (Z_1)^N \Rightarrow \underline{\underline{E = T^2\frac{\partial\ln Z}{\partial T} = NT}}$$

EQ. OF STATE: $\underline{\underline{p = \frac{\partial}{\partial A}(T\ln Z) = \frac{NT}{A}}}$