Problem III.2

We have treated the energy fluctuation in the canonical ensemble and the number of particles fluctuations in the grand canonical ensemble in class.

Following the usual procedure, we obtain

$$\underbrace{\langle(\Delta E)^2\rangle_g}_{\text{grand CA}} = -\left(\frac{\partial\langle E\rangle}{\partial\beta}\right)_{\mu,V} = T^2\left(\frac{\partial\langle E\rangle}{\partial T}\right)_{\mu,V}$$

For the number fluctuations the result was:

$$\langle(\Delta N)^2\rangle_g = T\left(\frac{\partial\langle N\rangle}{\partial\mu}\right)_{V,T} = \left(\frac{\partial\langle N\rangle}{\partial(\mu/T)}\right)_{V,T} = \left(\frac{\partial\langle N\rangle}{\partial\alpha}\right)_{V,T}, \quad \alpha = \frac{\mu}{T}.$$

Regarding E as a function of T and N, from

$$dE = \left(\frac{\partial E}{\partial T}\right)_N dT + \left(\frac{\partial E}{\partial N}\right)_T dN$$

we can easily obtain:

$$\left(\frac{\partial E}{\partial T}\right)_\alpha = \left(\frac{\partial E}{\partial T}\right)_N + \left(\frac{\partial E}{\partial N}\right)_T\left(\frac{\partial N}{\partial T}\right)_\alpha$$

Since $\langle(\Delta E)^2\rangle_{\underline{\underline{\text{CANONICAL}}}} = T^2\left(\frac{\partial E}{\partial T}\right)_N$, we have

$$\langle(\Delta E)^2\rangle - \langle(\Delta E)^2\rangle_{\text{CANONICAL}} = T^2\left(\frac{\partial E}{\partial N}\right)_T\left(\frac{\partial N}{\partial T}\right)_\alpha$$

$E \leftrightarrow \langle E\rangle$

To relate this to $\langle(\Delta N)^2\rangle$, consider

$d\Omega = p\,dV + S\,dT + N\,d\mu$. For a system with fixed volume ($dV = 0$) in terms of α: $d\mu = \alpha\,dT + T\,d\alpha \Rightarrow$

$$d\Omega = (S + \alpha N)\,dT + NT\,d\alpha$$

From this expression, for second derivatives:

$$\left(\frac{\partial(S+\alpha N)}{\partial\alpha}\right)_T = \left(\frac{\partial(NT)}{\partial T}\right)_\alpha = N + T\left(\frac{\partial N}{\partial T}\right)_\alpha$$

Problem III.2

Sheet 2

The expression ~~above~~ for the left hand side is obtained writing

$$dE = TdS - pdV + \mu dN \quad \text{as}$$

$$d(S + \alpha N) = \frac{1}{T}\, dE + N\, d\alpha$$

$$\Downarrow$$

$$\left(\frac{\partial (S + \alpha N)}{\partial \alpha}\right)_T = N + \frac{1}{T}\left(\frac{\partial E}{\partial \alpha}\right)_T$$

Thus ;

$$T^2\left(\frac{\partial N}{\partial T}\right)_\alpha = \left(\frac{\partial E}{\partial \alpha}\right)_T = \left(\frac{\partial E}{\partial N}\right)_T \left(\frac{\partial N}{\partial \alpha}\right)_T = \left(\frac{\partial E}{\partial N}\right)_T \langle(\Delta N)^2\rangle$$

$$\therefore \quad \underline{\langle(\Delta E)^2\rangle = \langle(\Delta E)^2\rangle_{\text{CANONICAL}} + \left[\left(\frac{\partial E}{\partial N}\right)_T\right]^2 \langle(\Delta N)^2\rangle}$$

$\square$