$$Y_{31}(\theta,\phi)\,Y_{20}(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} C_{\ell m}\,Y_{\ell m} \qquad [*]$$

(a) Multiply through by $Y^{\star}_{\ell' m'}$ and integrate over the unit sphere:

$$\begin{aligned}\int\!\!\int Y^{\star}_{\ell' m'}\,Y_{31}\,Y_{20}\,d^2\Omega &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} C_{\ell m}\int\!\!\int Y_{\ell' m'}Y_{\ell m}\,d^2\Omega \\ &= C_{\ell' m'}\end{aligned}$$

by orthonormality of the spherical harmonics. Examining the integrand, we see that the first selection rule implies that $m' = 1$; the second requires that $1 \le \ell' \le 5$; and so the third implies $\ell' = 1, 3, 5$ only. We conclude that the only nonzero coefficients are those with $(\ell', m') = (1,1), (3,1), (5,1)$.

(b) Of course, the coefficients can be calculated using "brute force" by evaluating the above integrals. But since there are so few nonzero coefficients, it is probably easier to determine the relevant linear combination Eq.([*]) algebraically. We have

$$\begin{aligned}Y_{51} &= N_{51}\,e^{\imath\phi}\,\sin\theta\,(21\cos^4\theta - 14\cos^2\theta + 1) \quad ; \quad N_{51} = -\frac{1}{16}\sqrt{\frac{165}{2\pi}} \\ Y_{31} &= N_{31}\,e^{\imath\phi}\,\sin\theta\,(5\cos^2\theta - 1) \quad ; \quad N_{31} = -\frac{1}{8}\sqrt{\frac{21}{\pi}} \\ Y_{11} &= N_{11}\,e^{\imath\phi}\,\sin\theta \quad ; \quad N_{11} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \\ Y_{20} &= N_{20}\,(3\cos^2\theta - 1) \quad ; \quad N_{20} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\end{aligned}$$

There follows

$$Y_{20}\,Y_{31} = N_{20}\,N_{31}\,e^{\imath\phi}\,\sin\theta\,(15\cos^4\theta - 8\cos^2\theta + 1)$$

Now,

$$15\cos^4\theta - 8\cos^2\theta + 1 = \frac{15}{21}\,(21\cos^4\theta - 14\cos^2\theta + 1) + \frac{2}{5}\,(5\cos^2\theta - 1) + \frac{24}{35}$$

so that

$$\begin{aligned}C_{51} &= \frac{15}{21}\,\frac{N_{20}\,N_{31}}{N_{51}} = \sqrt{\frac{25}{154\pi}} \approx 0.2273 \\ C_{31} &= \frac{2}{5}\,\frac{N_{20}\,N_{31}}{N_{31}} = \sqrt{\frac{1}{20\pi}} \approx 0.1262 \\ C_{11} &= \frac{24}{35}\,\frac{N_{20}\,N_{31}}{N_{11}} = \sqrt{\frac{9}{70\pi}} \approx 0.2023\end{aligned}$$