$$\nabla^2\psi + \frac{2mE}{\hbar^2} = 0 \qquad ; \qquad (\,|\vec{r}| \le a\,)$$

$$\begin{aligned}
\nabla^2\psi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \qquad ; \quad \text{since} \quad \psi(r,\theta,\phi) = \psi(r)
\end{aligned}$$

So we have

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + \frac{2mE}{\hbar^2}\psi = 0$$

and the boundary condition is that the wavefunction vanishes on the sphere, $\psi = 0$ for $r = a$.

This is already self-adjoint, and can be written in "standard form" (just multiply through by r^2):

$$\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + \frac{2mE}{\hbar^2}r^2\psi = 0$$

and we can just read off the coefficients:

$$p(x) = x^2 \qquad ; \qquad q(x) = 0 \qquad ; \qquad \lambda = \frac{2mE}{\hbar^2} \qquad ; \qquad w(x) = x^2$$

The bound on the lowest eigenvalue is, using $u(x) = 1 - (x/a)^2$ and so $u' = -2x/a^2$,

$$\lambda_0 \le \frac{\int_0^a x^2\left(\frac{4x^2}{a^4}\right)\,dx}{\int_0^a x^2\left[1-\left(\frac{x}{a}\right)^2\right]^2\,dx}$$

The numerator evaluates to $4a/5$ and the denominator works out to $8a^3/105$, so that

$$\lambda_0 \le \frac{21}{2}\,a^{-2} \qquad \Rightarrow \qquad E_0 \le \frac{21}{4}\frac{\hbar^2}{ma^2}$$

The exact result is only about 6% lower – not bad!