

$$M = \begin{pmatrix} 1 & 3 \\ 3 & b \end{pmatrix}$$

(a) Symmetric  $\Rightarrow M_{ij} = M_{ji}$ , which is true for any choice of  $b$ . Alternatively, with the understanding that standard terminology is to reserve “symmetric” to refer to real matrices, you might instead have (correctly) said that  $b$  can be any real number.

(b) Antisymmetric  $\Rightarrow M_{ij} = -M_{ji}$ , which is not possible for any choice of  $b$ .

(c) Orthogonal  $\Rightarrow \tilde{M} = M^{-1}$  (it's transpose equals its inverse). Thus, if  $M$  is orthogonal,  $M$  times  $\tilde{M}$  gives the identity matrix. But

$$M \cdot \tilde{M} = \begin{pmatrix} 10 & 3 + 3b \\ 3 + 3b & 9 + b^2 \end{pmatrix}$$

so  $M$  is not orthogonal for any  $b$ .

(d) Normal  $\Rightarrow [M, M^\dagger] = 0$ , i.e.  $M$  commutes with its complex-conjugate transpose. If  $b$  is real, this is clearly true since then  $M$  and  $M^\dagger$  are the same matrix. If  $b$  is complex, however,  $M$  is not normal since

$$M \cdot M^\dagger = \begin{pmatrix} 1 & 3 \\ 3 & b \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & b^* \end{pmatrix} = \begin{pmatrix} 10 & 3 + 3b \\ 3 + 3b & 9 + |b|^2 \end{pmatrix} \neq \begin{pmatrix} 10 & 3 + 3b \\ 3 + 3b^* & 9 + |b|^2 \end{pmatrix} = M^\dagger \cdot M$$

(e) Hermitian  $\Rightarrow M = M^\dagger$  which is true for any real value of  $b$ .

(f) Unitary  $\Rightarrow M^\dagger = M^{-1}$ . We can check by seeing if  $M$  times  $M^\dagger$  is the identity matrix. In part (d) we computed this product, and can see that  $M$  is not unitary for any value of  $b$ .

(g) For  $M = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ , the eigenvalues satisfy

$$\begin{vmatrix} 1 - \lambda & 3 \\ 3 & 1 - \lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - 2\lambda - 8 = 0 \quad \Rightarrow \quad \lambda = -2, 4$$

(h) The eigenvector for  $\lambda = -2$  satisfies

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -2 \begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad \begin{array}{l} x + 3y = -2x \\ 3x + y = -2y \end{array} \quad \Rightarrow \quad x = -y$$

so we can take the eigenvector to be any non-zero constant times  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

Meanwhile, since  $M$  is symmetric, the eigenvectors are mutually orthogonal, so the eigenvector corresponding to  $\lambda = 4$  is any (non-zero) constant times  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .