Evaluate : $\int_{-\infty}^{\infty} e^x \, \delta(3x^2-1)\, dx$

The function $g(x) = 3x^2-1$ has simple zeros at $x = \pm 1/\sqrt{3}$

Its slope $g'(x) = 6x = \pm 6/\sqrt{3}$ at these points.

So: $$\int_{-\infty}^{\infty} e^x \delta(3x^2-1)\, dx = \int_{-\infty}^{\infty} e^x \left\{ \frac{\delta(x-\frac{1}{\sqrt{3}})}{|6/\sqrt{3}|} + \frac{\delta(x+\frac{1}{\sqrt{3}})}{|-6/\sqrt{3}|} \right\} dx$$

$$= \frac{\sqrt{3}}{6} \int_{-\infty}^{\infty} e^x \left\{ \delta(x-\tfrac{1}{\sqrt{3}}) + \delta(x+\tfrac{1}{\sqrt{3}}) \right\} dx$$

$$= \boxed{\frac{\sqrt{3}}{6} \left\{ e^{1/\sqrt{3}} + e^{-1/\sqrt{3}} \right\}}$$

$$\approx 0.676$$

Let $f(x) = e^{-|x|}$. Find $f''(x)$.

There are a variety of approaches, ranging from very formal, to "look and see" numerics. Here's something in between:

First, it should be obvious that any "funny business" is confined to $x=0$. Staying away from this point (for now), we have

$(x>0)$: $f(x) = e^{-|x|} = e^{-x}$ so that $f''(x) = e^{-x} = e^{-|x|}$

$(x<0)$: $f(x) = e^{-|x|} = e^{+x}$ so that $f''(x) = e^{+x} = e^{-|x|}$

That is, $f''(x) = e^{-|x|}$ except (possibly) at $x=0$.

Next, examine $f(x)$ in the neighborhood of $x=0$:

$$f(x) = e^{-|x|} = 1 - |x| + \ldots$$

(the "..." terms vanish in the limit $|x| \to 0$)

$$\Rightarrow f''(x) = -|x|'' + \ldots$$

and in class we showed that $|x|'' = 2\delta(x)$, so $f'' \underset{(x\to 0)}{=} -2\delta(x)$

Putting this all together,

$$f''(x) = e^{-|x|} - 2\delta(x)$$

So that $f''(x) = f(x) + c\,\delta(x-x_0)$

with $\boxed{c = -2 \text{ and } x_0 = 0}$

Now a more formal derivation:

$$f(x) = e^{-|x|}$$

$$\Rightarrow \quad f'(x) = -|x|' e^{-|x|} = -|x|' f(x)$$

But $|x|' = \text{sgn}(x) = 2H-1$, so

$$f'(x) = -(2H-1) f(x). \quad [*]$$

$$\Rightarrow \quad f''(x) = -(2H-1) f'(x) - 2H' \cdot f(x)$$

$$= +(2H-1)^2 f(x) - 2\delta(x) f(x)$$

where I've used [*] in the 1^{st} term and $H' = \delta$ in the 2^{nd}.

But $2H-1 = \text{sgn}(x) \Rightarrow (2H-1)^2 = 1$

and $\delta(x) f(x) = \delta(x) f(0) = \delta(x)$ for this $f(x)$,

so

$$f''(x) = f(x) - 2\delta(x)$$

$$= f(x) + c\,\delta(x - x_0) \text{ with } c = -2 \text{ and } x_0 = 0.$$

(as before).