(a) Since $\vec{r} = \hat{x}\, a\cos\xi + \hat{y}\, b\sin\xi$, we have

$$d\vec{r} = -\hat{x}\, a\sin\xi\, d\xi + \hat{y}\, b\cos\xi\, d\xi$$

and so

$$\vec{r}\times d\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a\cos\xi & b\sin\xi & 0 \\ -a\sin\xi d\xi & b\cos\xi d\xi & 0 \end{vmatrix} = \hat{z}\, a b\, d\xi$$

$$\Rightarrow \quad \oint \vec{r}\times d\vec{r} = \int_{\xi=0}^{2\pi} \hat{z}\, a b\, d\xi = \hat{z}\, a b \cdot 2\pi$$

$$\therefore \quad \left|\oint \vec{r}\times d\vec{r}\right| = 2\pi a b \quad (\checkmark)$$

(b) In general, $\oint d\vec{\lambda}\times\vec{P} = \iint_S (d\vec{\sigma}\times\nabla)\times\vec{P}$

For a planar curve, $d\vec{\sigma} = \hat{z}\, d\sigma$, and letting $\vec{P} = \vec{r}$ and $d\vec{\lambda} = d\vec{r}$, the general formula reduces to:

$$\oint d\vec{r}\times\vec{r} = \iint_S d\sigma\,(\hat{z}\times\nabla)\times\vec{r} \qquad [*]$$

But: $$(\hat{z}\times\nabla) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ \partial_x & \partial_y & \partial_z \end{vmatrix} = \hat{y}\,\partial_x - \hat{x}\,\partial_y$$

so that $$(\hat{z}\times\nabla)\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\partial_y & \partial_x & 0 \\ x & y & z \end{vmatrix} = \ldots = -2\hat{z}$$

Use this on [*]:

$$\oint d\vec{r}\times\vec{r} = \iint_S d\sigma\cdot(-2\hat{z}) = -2\hat{z}\iint_S d\sigma = -2\hat{z}\,A$$

$A \equiv$ "area enclosed"

$$\Rightarrow \left|\oint \vec{r}\times d\vec{r}\right| = 2A \quad (\checkmark)$$

$\boxtimes$