$$\left.\begin{aligned} x &= \tfrac{1}{2}(u^2 - v^2) \\ y &= uv \\ z &= w \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} dx &= u\,du - v\,dv \\ dy &= u\,dv + v\,du \\ dz &= dw \end{aligned}\right.$$

(a) Compute $ds^2 = dx^2 + dy^2 + dz^2$

$$= (u\,du - v\,dv)^2 + (u\,dv + v\,du)^2 + dw^2$$

$$= \cdots$$

$$= (u^2+v^2)\,du^2 + (u^2+v^2)\,dv^2 + dw^2$$

(orthogonal ✓)

(b) $ds^2 = (h_u\,du)^2 + (h_v\,dv)^2 + (h_w\,dw)^2$

$$\Rightarrow \quad h_u = h_v = \sqrt{u^2+v^2} \quad \text{and} \quad h_w = 1$$

(c) In general (cf. Arfken & Weber inside front cover)

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + (\text{cyclic } 1 \to 2 \to 3 \to 1)\right]$$

$$= \frac{1}{u^2+v^2}\left[\frac{\partial}{\partial u}\left(\frac{\sqrt{\cdots}\cdot 1}{\sqrt{\cdots}}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1\cdot\sqrt{\cdots}}{\sqrt{\cdots}}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{\sqrt{\ }^{2}}{1}\frac{\partial f}{\partial w}\right)\right]$$

$$= \frac{1}{u^2+v^2}\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right) + \frac{\partial^2 f}{\partial w^2}$$

$(\longrightarrow)$

(d) The specified area $A = \iint d\sigma_{uv}$

where $d\sigma_{uv}$ is the area element lying in plane spanned by u and v (and, so, normal to $\hat{w}$). Now, in general

$$d\sigma_{ij} = h_i h_j \, dq_i \, dq_j \qquad \text{(cf. Arfken \& Weber Eq (2.10))}$$

So that, in this example,

$$d\sigma = \left(\sqrt{u^2+v^2}\right)^2 du\, dv$$

$$\Rightarrow \quad A = \int_{u=0}^{2} \int_{v=1}^{2} (u^2+v^2)\, du\, dv$$

$$= \int_{u=0}^{2} \int_{v=1}^{2} u^2 du\, dv + \int_{u=0}^{2} \int_{v=1}^{2} v^2 du\, dv$$

$$= \left[\frac{u^3}{3}\right]_0^2 \cdot 1 + 2 \cdot \left[\frac{v^3}{3}\right]_1^2$$

$$= \frac{8}{3} + 2 \cdot \frac{7}{3}$$

$$= \boxed{\frac{22}{3}}$$

<u>NOTE</u>: This particular coordinate system has a name: "Parabolic Cylindrical Coordinates"