1'

Prove $$\varepsilon_{ijk}\,\varepsilon_{ipq} = \delta_{jp}\,\delta_{kq} - \delta_{jq}\,\delta_{kp} \qquad [*]$$

Since each free index $jkpq$ can independently be 1, 2, 3, there are $3^4 = 81$ cases to check. It is reasonable to write a computer program to run through these 81 cases and compare the two expressions.

With some thought, we can prove the relation by systematically reducing the set of possibilities. First, if $j=k$ $\varepsilon_{ijk}=0$ so the LHS $=0$; it is obvious that setting $j=k$ makes the two products on the RHS identical, so RHS $=0$ (✓). The same reasoning shows that if $p=q$, LHS $=0=$ RHS (✓).

That leaves the cases with ($j \neq k$ and $p \neq q$) to test; $6 \times 6 = 36$ cases. Write out the LHS:

$$\varepsilon_{ijk}\,\varepsilon_{ipq} = \varepsilon_{1jk}\,\varepsilon_{1pq} + \varepsilon_{2jk}\,\varepsilon_{2pq} + \varepsilon_{3jk}\,\varepsilon_{3pq} \qquad [**]$$

For a given choice of (jk), only 1 of these 3 terms can be non-zero, and that requires that (pq) consist of the same two integers. That is:

LHS $=0$ unless ($j=p$ and $k=q$) OR ($j=q$ and $k=p$)

Looking at the RHS of [*] it is evident that RHS $=0$ as well, unless one of the two conditions holds (✓)

Now, if $j=p$ and $k=q$, the two factors in the nonzero term on the rightside of [**] are the same — either $+1$ or -1 — so their product is $+1$. The RHS of [*] also equals $+1$ for this case. (✓)

Finally, if $j=q$ and $k=p$, the two factors in the nonzero term on the right side of [**] are opposite — one is $+1$, the other -1 — so their product is -1, in agreement with the RHS of [*] (✓). $\square$

(2)

Consider the first line of Dr. Diligent's "proof":

$$\left\{\nabla \times (\vec{A} \times \vec{B})\right\}_i = \varepsilon_{ijk}\, \partial_j\, \varepsilon_{klm} A_l B_m$$

This is OK <u>provided</u> we understand that ∂_j operates on everything to its right. If Diligent understands this, he forgets it when he comes to the last step, where he writes

$$\partial_j B_j A_i - \partial_j A_j B_i = \left\{(\vec{\nabla}\cdot\vec{B})\vec{A} - (\nabla\cdot\vec{A})\vec{B}\right\}_i$$

$\times$

The corrected proof, starting at this point, is

$$\partial_j B_j A_i - \partial_j A_j B_i$$

$$= (\partial_j B_j) A_i + B_j(\partial_j A_i) - (\partial_j A_j) B_i - A_j(\partial_j B_i)$$

$$= (\nabla\cdot\vec{B})A_i + \vec{B}\cdot\nabla A_i - (\nabla\cdot\vec{A})B_i - \vec{A}\cdot\nabla B_i$$

$$= \left\{(\vec{\nabla}\cdot\vec{B})\vec{A} + \vec{B}\cdot\nabla\vec{A} - (\nabla\cdot\vec{A})\vec{B} - \vec{A}\cdot\nabla\vec{B}\right\}_i$$

This correct identity can be found on the inside front cover of our textbook.

$\square$