In this problem, you are invited to estimate the ground state energy of a particle-in-a-box, when the box is an impenetrable sphere of radius a. The Schrodinger Equation for an energy eigenfuction $\psi(\vec{r})$ reads

$$\nabla^2\psi + \frac{2mE}{\hbar^2}\,\psi = 0 \qquad ; \qquad (\,|\vec{r}| \leq a\,)$$

where m is the particle mass, E is the energy, and $\hbar$ is Planck's constant divided by 2π. The wavefunction is identically zero everywhere outside the sphere.

(a) Use the fact that the ground state is spherically symmetric to verify that the above equation reduces to the Sturm-Liouville problem

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + \frac{2mE}{\hbar^2}\psi = 0$$

with $0 \leq r \leq a$ and boundary condition $\psi(r = a) = 0$.

(b) Use the trial wavefunction $\psi = 1 - \left(\frac{r}{a}\right)^2$ to get an estimate for the ground state energy. This yields a rigorous upper bound to the exact result $\pi^2\hbar^2/(2ma^2)$.

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Recall that the variational functional is

$$F\left[u(x)\right] = \frac{\int_a^b \left\{p\,(u')^2 - qu^2\right\}\,dx}{\int_a^b w\,u^2\,dx}$$

where p, q, w are the usual Sturm-Liouville functions.