Suppose one wishes to express the product $Y_{31}(\theta,\phi)\,Y_{20}(\theta,\phi)$ as a linear combination of spherical harmonics, i.e.

$$Y_{31}(\theta,\phi)\,Y_{20}(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} C_{\ell m}\,Y_{\ell m} \qquad [*]$$

where the $C_{\ell m}$ are numerical constants to be determined.

(a) Explain how you can use the general selection rules for the product of 3 spherical harmonics

$$\int\!\!\int Y^{\star}_{\ell_1 m_1}\,Y_{\ell_2 m_2}\,Y_{\ell_3 m_3}\,d^2\Omega \;=\; 0 \qquad \text{unless}$$

$$\begin{aligned} &(i) && m_2 + m_3 = m_1; \\ &(ii) && |\ell_2 - \ell_3| \le \ell_1 \le |\ell_2 + \ell_3|; \\ \text{and}\quad &(iii) && \ell_1 + \ell_2 + \ell_3 = \text{even integer}; \end{aligned}$$

to show that only 3 terms in the expansion $[\star]$ are non-zero.

(b) Explicitly evaluate these nonzero coefficients $C_{\ell m}$.