Legendre Polynomials

Associated Legendre Functions

Spherical Harmonics

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + \ell\left(\ell-1\right)\Theta = 0$$

$$x = \cos\theta \qquad ; \qquad \Theta \to P$$

The Associated Legendre Equation:

$$\left(1-x^2\right)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \left[\ell\left(\ell-1\right) - \frac{m^2}{1-x^2}\right]P = 0 \qquad ; \qquad x \in [-1,1]$$

$P_\ell^m(x)$ are the Associated Legendre Functions

$P_n(x) = P_{\ell-1}^0(x)$ are the Legendre Polynomials

Spherical Harmonics:

$$Y_{\ell m}(\theta,\phi) = \mathcal{N}_{\ell m}\,\Theta_\ell^m(\theta)\,e^{\pm im\phi}$$

Legendre Polynomials

Properties of Legendre Polynomials $P_n(x)$:

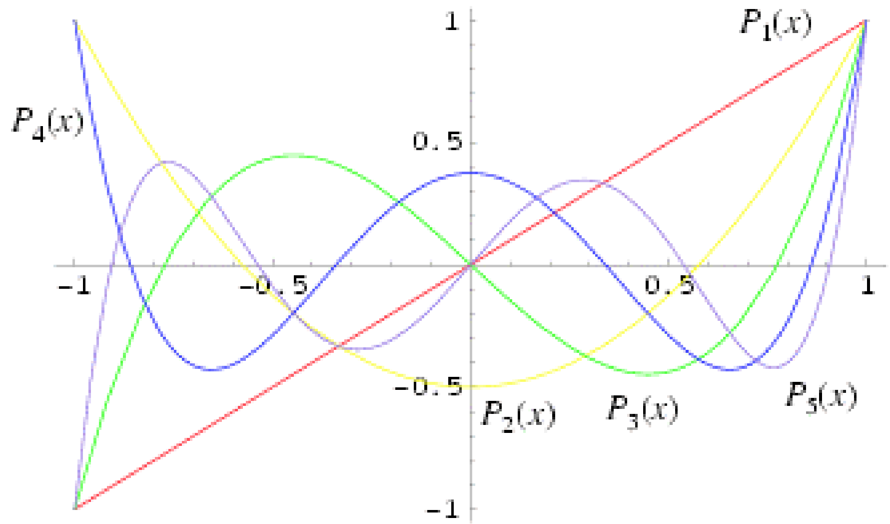
Rodrigues' Formula

$$P_n(x) = \frac{1}{2^n n!} \left(\frac{d}{dx} \right)^n \left(x^2 - 1 \right)^n$$

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{3}{2}x^2 - \frac{1}{2} \\ &etc. \end{aligned}$$

Generating Function

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n \qquad ; \qquad |t| < 1$$



The first few Legendre polynomials are

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$
$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$
$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5).$$

Associated Legendre Functions

Legendre's Equation is Sturm-Liouville self-adjoint:

$$\frac{d}{dx}\left[\left(1-x^2\right)P_n'(x)\right]+n(n+1)P_n(x)=0 \qquad ; \qquad x\in[-1,1]$$

weight function	$w(x)=1$
eigenvalues	$\lambda_n=n(n+1)$

The eigenfunctions are:

orthogonal $$\int_{-1}^{1}P_n(x)P_m(x)\,dx=0 \qquad ; \qquad m\neq n$$

complete $$f(x)=\sum_{n=0}^{\infty}a_n P_n(x) \qquad ; \qquad x\in[-1,1]$$

$\Rightarrow$

"Legendre Series" $$a_m=\frac{2m+1}{2}\int_{-1}1f(x)P_m(x)\,dx$$

Recurrence relations, e.g

$$P_{n+1}'(x)-P_{n-1}'(x) = (2n+1)P_n(x)$$

$$P_{n+1}'(x) = n(n+1)P_n(x)+x\,P_n'(x)$$

The Associated Legendre Equation:

$$\left(1-x^2\right)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+\left[\ell\left(\ell-1\right)-\frac{m^2}{1-x^2}\right]P=0 \qquad ; \qquad x\in[-1,1]$$

Properties of Associated Legendre functions $P_n^m(x)$ $\quad (n=1,2,\ldots;m=1,2,\ldots,n)$:

$$P_n^m(x)=\left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}P_n(x)$$

NOTE: $\quad P_n^m(x=\pm 1)=0 \qquad (m>0)$

$$P_0^0(x) = 1$$
$$P_1^0(x) = x$$
$$P_1^1(x) = -(1-x^2)^{1/2}$$
$$P_2^0(x) = \frac{1}{2}(3x^2-1)$$
$$P_2^1(x) = -3x(1-x^2)^{1/2}$$
$$P_2^2(x) = 3(1-x^2)$$
$$P_3^0(x) = \frac{1}{2}x(5x^2-3)$$
$$P_3^1(x) = \frac{3}{2}(1-5x^2)(1-x^2)^{1/2}$$
$$P_3^2(x) = 15x(1-x^2)$$
$$P_3^3(x) = -15(1-x^2)^{3/2}$$
$$P_4^0(x) = \frac{1}{8}(35x^4-30x^2+3)$$
$$P_4^1(x) = \frac{5}{2}x(3-7x^2)(1-x^2)^{1/2}$$
$$P_4^2(x) = \frac{15}{2}(7x^2-1)(1-x^2)$$
$$P_4^3(x) = -105x(1-x^2)^{3/2}$$
$$P_4^4(x) = 105(1-x^2)^2$$
$$P_5^0(x) = \frac{1}{8}x(63x^4-70x^2+15).$$

$$P_0^0(\cos\theta) = 1$$
$$P_1^0(\cos\theta) = \cos\theta$$
$$P_1^1(\cos\theta) = -\sin\theta$$
$$P_2^0(\cos\theta) = \frac{1}{2}(3\cos^2\theta-1)$$
$$P_2^1(\cos\theta) = -3\sin\theta\cos\theta$$
$$P_2^2(\cos\theta) = 3\sin^2\theta$$
$$P_3^0(\cos\theta) = \frac{1}{2}\cos\theta(5\cos^2\theta-3)$$
$$P_3^1(\cos\theta) = -\frac{3}{2}(5\cos^2\theta-1)\sin\theta$$
$$P_3^2(\cos\theta) = 15\cos\theta\sin^2\theta$$
$$P_3^3(\cos\theta) = -15\sin^3\theta.$$

Recurrence relations, e.g.

$$(2n+1)xP_n^m = (n+m)P_{n-1}^m + (n-m+1)P_{n+1}^m$$

Orthogonality formulas:

$$\int_{-1}^{1} P_p^m(x)\, P_q^m(x)\, dx = \frac{2}{2q+1}\, \frac{(q+m)!}{(q-m)!}\, \delta_{p,q}$$

$$\int_{-1}^{1} P_n^m(x)\, P_n^k(x)\, \left(1-x^2\right)^{-1}\, dx = \frac{(n+m)!}{m(n-m)!}\, \delta_{m,k}$$

Spherical Harmonics

Properties of Spherical Harmonics (Condon-Shortley Phase)

$$Y_{\ell m}(\theta,\phi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta) e^{im\phi}$$

$\ell = 0, 1, 2, \ldots \quad ; \quad m = -\ell, \ldots, \ell.$

Complete set of functions on the sphere:

$$f(\theta,\phi) = \sum_{\ell,m} a_{\ell m} Y_{\ell m}(\theta, phi)$$

Orthogonality relation:

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} Y^\star_{\ell_1 m_1}(\theta,\phi) Y_{\ell_2 m_2}(\theta,\phi) \sin\theta \, d\theta \, d\phi = \delta_{\ell_1,\ell_2}\, \delta_{m_1,m_2}$$

Addition Theorem:

$$P_n(\cos\gamma) = \frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y_{nm}(\theta_1,\phi_1) Y^\star_{nm}(\theta_2,\phi_2)$$

where $\gamma \angle \{(\theta_1,\phi_1),(\theta_2,\phi_2)\}$

The first several spherical harmonics

$$Y_0^0(\theta,\phi) = \frac{1}{2}\frac{1}{\sqrt{\pi}}$$

$$Y_1^{-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{-i\phi}$$

$$Y_1^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$$

$$Y_1^1(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^{-2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{-2i\phi}$$

$$Y_2^{-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta\, e^{-i\phi}$$

$$Y_2^0(\theta,\phi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1)$$

$$Y_2^1(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_3^{-3}(\theta,\phi) = \frac{1}{8}\sqrt{\frac{35}{\pi}}\sin^3\theta\, e^{-3i\phi}$$

$$Y_3^{-2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta\cos\theta\, e^{-2i\phi}$$

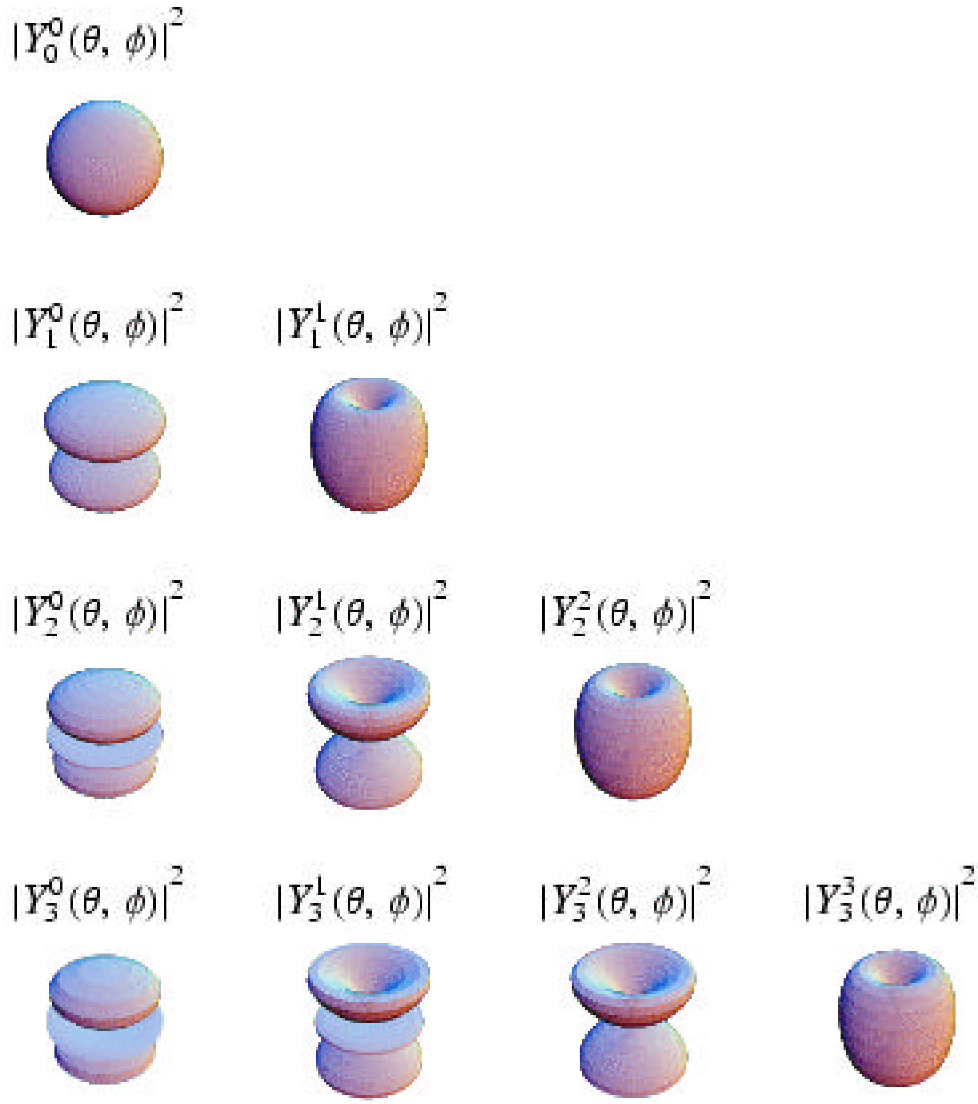
$$Y_3^{-1}(\theta,\phi) = \frac{1}{8}\sqrt{\frac{21}{\pi}}\sin\theta(5\cos^2\theta - 1)\, e^{-i\phi}$$

$$Y_3^0(\theta,\phi) = \frac{1}{4}\sqrt{\frac{7}{\pi}}(5\cos^3\theta - 3\cos\theta)$$

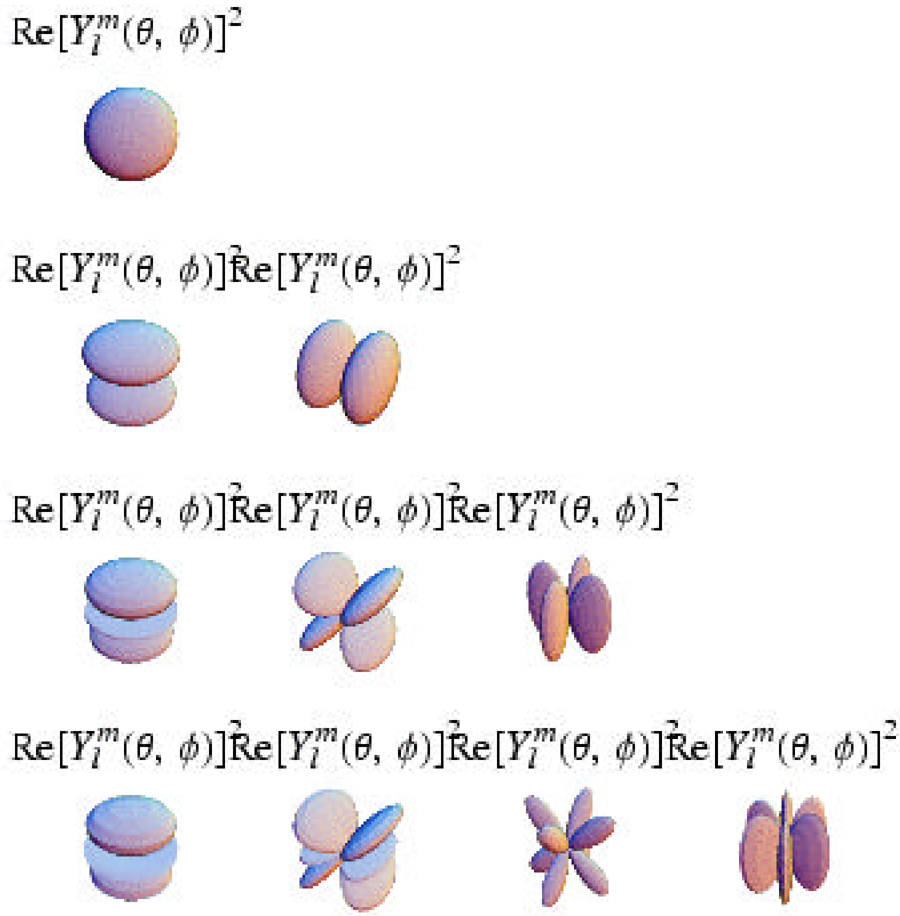
$$Y_3^1(\theta,\phi) = -\frac{1}{8}\sqrt{\frac{21}{\pi}}\sin\theta(5\cos^2\theta - 1)\, e^{i\phi}$$

$$Y_3^2(\theta,\phi) = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta\cos\theta\, e^{2i\phi}$$

$$Y_3^3(\theta,\phi) = -\frac{1}{8}\sqrt{\frac{35}{\pi}}\sin^3\theta\, e^{3i\phi}.$$

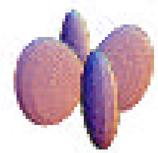


http://mathworld.wolfram.com/SphericalHarmonic.html


$\mathrm{Re}[Y_l^m(\theta, \phi)]^2$
$\mathrm{Re}[Y_l^m(\theta, \phi)]^2$ $\mathrm{Re}[Y_l^m(\theta, \phi)]^2$
$\mathrm{Re}[Y_l^m(\theta, \phi)]^2$ $\mathrm{Re}[Y_l^m(\theta, \phi)]^2$ $\mathrm{Re}[Y_l^m(\theta, \phi)]^2$
$\mathrm{Re}[Y_l^m(\theta, \phi)]^2$ $\mathrm{Re}[Y_l^m(\theta, \phi)]^2$ $\mathrm{Re}[Y_l^m(\theta, \phi)]^2$ $\mathrm{Re}[Y_l^m(\theta, \phi)]^2$

$\mathrm{Im}[Y_l^m(\theta, \phi)]^2$

$\mathrm{Im}[Y_l^m(\theta, \phi)]^2$ $\mathrm{Im}[Y_l^m(\theta, \phi)]^2$

2

$\mathrm{Im}[Y_l^m(\theta, \phi)]^2$ $\mathrm{Im}[Y_l^m(\theta, \phi)]^2$ $\mathrm{Im}[Y_l^m(\theta, \phi)]^2$