Bessel Functions

first kind, second kind, Neumann, Hankel

Bessel's Differential Equation:

$$x^2\Psi'' + x\Psi' \left(x^2 - \nu^2\right) \Psi = 0$$

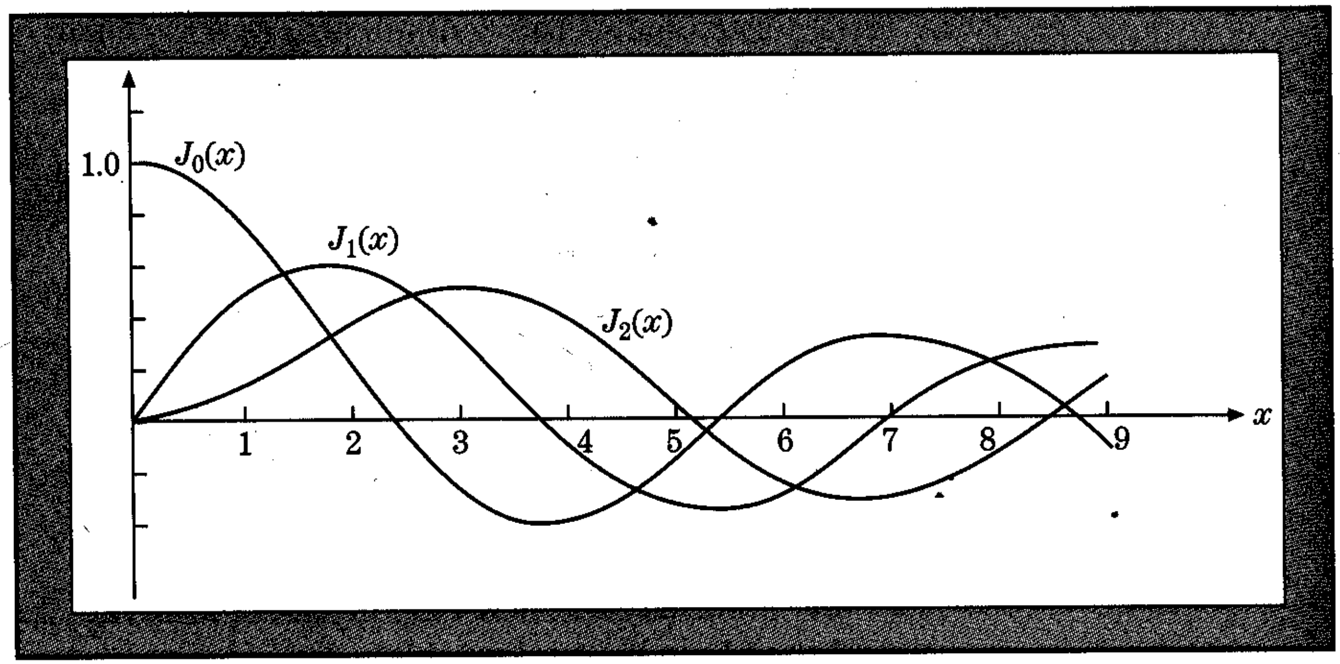
has two linearly independent solutions $J_\nu(x)$ *and* $Y_\nu(x)$:

$$J_\nu(x) = \sum_{s=0}^{\infty} \frac{(-1)^2}{s!\,(\nu + s)!} \left(\frac{x}{2}\right)^{\nu+2s}$$

$$Y_\nu(x) = \frac{\cos(\nu\pi)\; J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$$

Figure 12.1

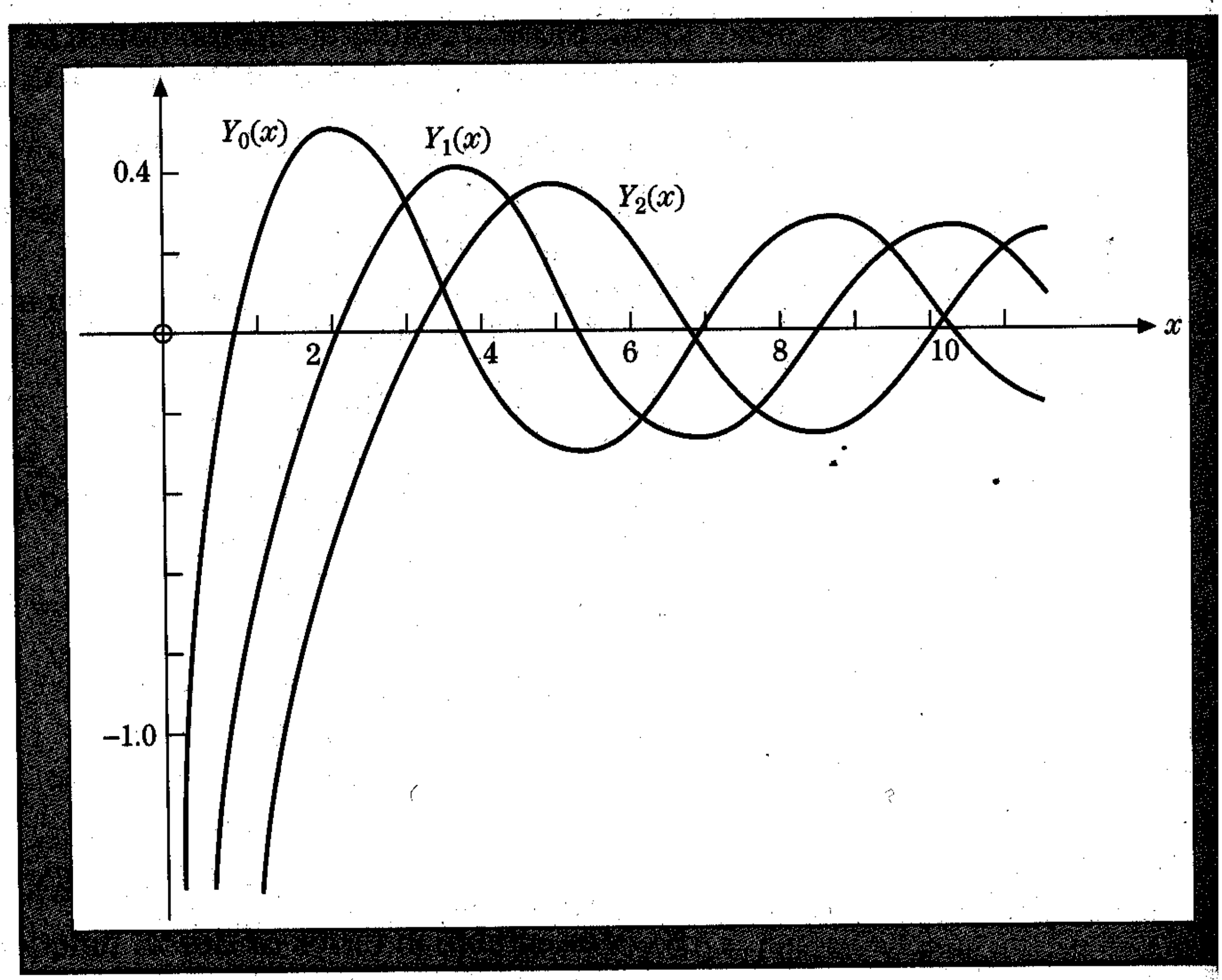
Bessel Functions, $J_0(x)$, $J_1(x)$, and $J_2(x)$



Source: Weber & Arfkin

Figure 12.7

Neumann Functions
$Y_0(x)$, $Y_1(x)$, and $Y_2(x)$



Source: Weber & Arfkin

(A Few) Properties of Bessel Functions

generating function

$$e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

addition theorem

$$J_m(u+v) = \sum_{\ell=-\infty}^{\infty} J_\ell(u)\, J_{m-\ell}(v)$$

integral representations

$$J_n(x) = \begin{cases} \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\ \cos n\theta\ d\theta & ; \quad n = 0, 2, 4, \ldots \\ \frac{1}{\pi}\int_0^\pi \sin(x\sin\theta)\ \sin n\theta\ d\theta & ; \quad n = 1, 3, 5, \ldots \end{cases}$$

$$Y_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh t)\ dt \qquad ; \qquad x > 0$$

orthogonality over the interval $\rho \in [0,a]$

$$\int_0^a J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\, J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\,\rho\ d\rho = 0 \qquad ; \qquad m \neq n$$

completeness (Bessel Series) $\rho \in [0,a]$:

$$f(\rho) = \sum_{m=1}^{\infty} C_{\nu m}\, J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right) \qquad ; \qquad \nu > -1$$

$$\text{where} \qquad C_{\nu m} = \frac{2}{a^2\left[J_{\nu+1}\left(\alpha_{\nu m}\right)\right]^2}\int_0^a f(\rho)\, J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\,\rho\ d\rho$$

closure

$$\int_0^\infty J_\nu(\alpha\rho)\, J_\nu(\alpha'\rho)\,\rho\ d\rho = \frac{1}{\alpha}\,\delta(\alpha-\alpha') \qquad ; \qquad \nu > -\frac{1}{2}$$

Fourier Series Expansions

$$\cos(x\sin\theta) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\ \cos(2n\theta)$$

$$\sin(x\sin\theta) = 2\sum_{n=1}^{\infty} J_{2n-1}(x)\ \sin\left[(2n-1)\,\theta)\right]$$

Recurrence relations

$$\begin{aligned}
J_{n-1}(x) + J_{n+1}(x) &= \frac{2n}{x} J_n'(x) \\
J_{n-1}(x) - J_{n+1}(x) &= 2J_n'(x) \\
J_{n-1}(x) - \frac{n}{x} J_n(x) &= J_n'(x) \\
\frac{d}{dx}\left[x^n J_n(x)\right] &= x^n J_{n-1}(x)
\end{aligned}$$

Worked Examples:

- Normal Modes of an annular drumhead
- Green function for a Bessel equation
- Diffraction: optical resolution & Rayleigh's criterion
- Nonlinear dynamics of a SQUID

Alone on a desert island . . .

Use: $$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

and $$J_0(x) + 2\sum_{m=1}^{\infty} J_{2m}(x) = 1$$

Pick (say) $\overline{J}_{12}(x) = 10,586$ (Dow Jones?) and $\overline{J}_{11} = 3.1415$ (π), and then

$$\overline{J}_{n-1} = -\overline{J}_{n+1}(x) + \frac{2n}{x}\overline{J}_n(x)$$

to generate $\overline{J}_n$ down to $n = 0$, and normalize (i.e. divide through) by N:

$$N = \overline{J}_0(x) + 2\sum_{m=1}^{\infty} \overline{J}_{2m}(x) = 1$$

n	$\overline{J}_n(1)$	$\overline{J}_n(1)/N$	$J_n(1)^{exact}$
0	$-3.0662e+013$	0.7652	0.7652
1	$-1.7633e+013$	0.44005	0.44005
2	$-4.6043e+012$	0.1149	0.1149
3	$-7.8392e+011$	0.019563	0.019563
4	$-9.9241e+010$	0.0024766	0.0024766
5	$-1.0008e+010$	0.00024976	0.00024976
6	$-8.3902e+008$	$2.0938e-005$	$2.0938e-005$
7	$-6.0200e+007$	$1.5023e-006$	$1.5023e-006$
8	$-3.7756e+006$	$9.4223e-008$	$9.4223e-008$
9	$-2.1034e+005$	$5.2492e-009$	$5.2493e-009$
10	-10517	$2.6246e-010$	$2.6306e-010$
11	3.1415	$-7.8399e-014$	$1.198e-011$
12	10586	$-2.6418e-010$	$4.9997e-013$

n	$\overline{J}_n(1/2)$	$\overline{J}_n(1/2)/N$	$J_n(1/2)^{exact}$
0	$-3.7543e+016$	0.93847	0.93847
1	$-9.6917e+015$	0.24227	0.24227
2	$-1.2243e+015$	0.030604	0.030604
3	$-1.0256e+014$	0.0025637	0.0025637
4	$-6.4301e+012$	0.00016074	0.00016074
5	$-3.2218e+011$	$8.0536e-006$	$8.0536e-006$
6	$-1.3444e+010$	$3.3607e-007$	$3.3607e-007$
7	$-4.8068e+008$	$1.2016e-008$	$1.2016e-008$
8	$-1.5034e+007$	$3.7582e-010$	$3.7582e-010$
9	$-4.1791e+005$	$1.0447e-011$	$1.0447e-011$
10	-10448	$2.6117e-013$	$2.6132e-013$
11	3.1415	$-7.8529e-017$	$5.9419e-015$
12	10586	$-2.6462e-013$	$1.2384e-016$